A1 **Logic** (25 points)

$\models$ is the (semantic) *consequence* relation for a first order predicate logic language ℓ adequate to express the formulas of this problem.[1]

a. (9 points)

Let

$$F_1 = \forall x R(x,x), \tag{1}$$

$$F_2 = \forall x \forall y (R(x,y) \to R(y,x)), \ \textit{and} \tag{2}$$

$$F_3 = \forall x \forall y \forall z ((R(x,y) \ \wedge \ R(y,z)) \to R(x,z)). \tag{3}$$

For *each* of the following *dis*prove it by exhibiting a suitable structure (different structures for different disproofs).

a(i). (3 points) $\{F_1, F_2\} \models F_3$.

a(ii). (3 points) $\{F_1, F_3\} \models F_2$.

a(iii). (3 points) $\{F_2, F_3\} \models F_1$.

You need not verify your structures work; just make sure they do.

b. (8 points)

Let

$$G_1 = \forall x \forall y (R(x,y) \to \neg R(y,x)), \tag{4}$$

$$G_2 = \forall x \forall y \forall z ((R(x,y) \ \wedge \ R(y,z)) \to R(x,z)), \tag{5}$$

$$G_3 = \forall x \exists y R(x,y), \ \textit{and} \tag{6}$$

$$G_4 = \neg \exists x \forall y R(y,x). \tag{7}$$

Prove the following step-by-step using resolution or another sound and complete technique of your **stated** *choice.*[2]

$$\{G_1, G_2, G_3\} \models G_4. \tag{8}$$

c. (8 points)

Prove *by resolution* (showing all your steps) the following.

For credit you must do this one employing resolution.

$\{\forall x \forall y (H(y) \to A(x,y)), \forall x \forall y (F(x) \to A(x,y)), \forall x (\neg H(x) \to F(x))\} \models \exists x \exists y (A(x,y) \ \wedge \ A(y,x))$.

[1] If $(\Gamma \cup \{A\})$ is a set of formulas of ℓ, $\Gamma \models A$ means that every model of Γ satisfies A.

[2] The technique you employ could be, for *example*, from a resolution system, one of many tableaux systems, a Hilbert style system, a Gentzen style system, ...

A2 **Logic** (25 points)

First: translate each of the four sentences a through d inclusive below into a suitable first order predicate logic language. Of course you should *first specify that language* and *define an interpretation of its symbols* of relevance to your translation.

Your specified language should include *among other things* symbols of the appropriate types (you must say *which* types) for expressing 'drinks blood', 'is happy', 'is a child of', 'can bite'. You *may, if you wish*, make the universe/domain of discourse the set of all bats and thereby avoid having to explicitly provide a symbol for expressing 'is a bat'.

Next: work part e below.

a. (5 points)
   Every bat which drinks blood is happy if all its children can bite.

b. (5 points)
   Bats which drink blood can bite.

c. (5 points)
   Any bat drinks blood if it's the child of at least one blood drinking bat.

d. (5 points)
   All blood drinking bats are happy.

e. (5 points)
   For *this* part, e, write the following argument in the notation from first order logic. It is *possible* to get full credit (5 points) for this part, e, even if you plug in your answers from a through d inclusive and some of them are not quite right. N.B. You need not prove or disprove the argument — just express it correctly.

   The argument: Sentence d is a logical consequence of sentences a through c inclusive.

A3 **Logic** (25 points)

Let $\mathcal{I}$ be an interpretation (or structure) for a predicate logic formula F with corresponding universe of discourse $U_{\mathcal{I}}$. For *example*: for each variable x in F, $\mathcal{I}(x) \in U_{\mathcal{I}}$; for each n-ary predicate symbol P in F, $\mathcal{I}(P) \subseteq U_{\mathcal{I}}^n$; for each n-ary function symbol f in F, $\mathcal{I}(f) : U_{\mathcal{I}}^n \rightarrow U_{\mathcal{I}}$; $\ldots$.

Suppose s, t are terms built from the constituents of F and x is a variable from F and that G is a formula built from the constituents of F.

Then we write

$$\{s\}[x/t] \tag{9}$$

to mean the result of simultaneously substituting for each occurrence of x in the term s, the term t.

We write

$$\{G\}[x/t] \tag{10}$$

to mean the result of simultaneously substituting for each *free* occurrence of x in the formula G, the term t.

Suppose $u \in U_{\mathcal{I}}$.

Then we write

$$\mathcal{I}[x/u] \tag{11}$$

to mean the variant of the interpretation $\mathcal{I}$ which is just like $\mathcal{I}$ *except* that $\mathcal{I}[x/u]$ interprets x to mean u, i.e., $\mathcal{I}[x/u](x) \overset{\text{def}}{=} u$; whereas, $\mathcal{I}(x)$ may or may not $= u$.

a. (6.25 points)

Prove by mathematical induction on the logical complexity of the term s that

$$\mathcal{I}[x/\mathcal{I}(t)](s) = \mathcal{I}(\{s\}[x/t]). \tag{12}$$

b. (12.50 points)

N.B. For *this* part of A3 you may use without proof the result from A3 part a.

Definition *t is free for x in F* $\overset{\text{def}}{\Leftrightarrow}$ for *no* variable y in t does x occur free within any subformula F' of F, *where* F' is either of the form $(\forall y)H$ or of the form $(\exists y)H$.

Example 1 The term $f(x, y)$ *is* free for x in the formula $(\exists z)P(x, z)$.

Example 2 The term $f(x, y)$ is *not* free for x in the formula $(\exists y)P(x, y)$.

Suppose t is free for x in F. Then: *prove by mathematical induction* on the logical complexity of the formula F that

$$\mathcal{I}[x/\mathcal{I}(t)](F) = \mathcal{I}(\{F\}[x/t]). \tag{13}$$

N.B. *Be sure that your proof makes it very clear how you are using the hypothesis that t is free for x in F.*

c. (6.25 points)

Explicitly present an example t, x, and F for which *both*

- t is *not* free for x in F *and*
- (13) above *fails.*

N.B. *You need not show your t, x, and F work, just make sure they do.*

A4 **Logic** (25 points)

For *this* problem, A4, we provide *considerable* information, much of which is to *motivate* the problem (which is to prove a surprising result). *Then*, near the end and on the next page, we state what is to be done for this problem and subsequently give a hint making it not so difficult. (◡̈)

Definition PA *(called:* the f.o. theory of Peano arithmetic*) has binary predicate symbol* $=$*, binary function symbols* $+, \cdot$*, constant symbols* $0, 1, 2, \ldots$*, and the equality axioms together with the infinite, algorithmically decidable set of the following axioms (in infix notation):*

$$(\forall v_1) v_1 + 0 = v_1; \tag{14}$$

$$(\forall v_1)(\forall v_2) v_1 + (v_2 + 1) = (v_1 + v_2) + 1; \tag{15}$$

$$(\forall v_1) v_1 \cdot 0 = 0; \tag{16}$$

$$(\forall v_1)(\forall v_2) v_1 \cdot (v_2 + 1) = (v_1 \cdot v_2) + v_1; \tag{17}$$

$$0 + 1 = 1; \tag{18}$$

for each positive $n \in N$*,*

$$\underbrace{1 + \ldots + 1}_{n\ 1\text{'s}} = n, \tag{19}$$

where the n *on the right-hand side of (19) is understood to be the constant symbol for representing* n *and where it is again understood that the expression* $1 + \ldots + 1$ *is parenthesized with association to the left;*

$$(\forall v_1)(\forall v_2)[v_1 + 1 = v_2 + 1 \supset v_1 = v_2]; \tag{20}$$

$$(\forall v_1) 0 \neq v_1 + 1; \tag{21}$$

and all cwffs (closed well formed formulas) of the the underlying language of the form

$$\text{closure}([[\{A\}[x/0] \ \wedge \ (x)[A \supset \{A\}[x/(x+1)]] \supset (x)A]), \tag{22}$$

where A *is an arbitrary wff (well formed formula) of PA,* where *closure*(B) *is* $(\forall x_1) \cdots (\forall x_m) B$ *for the distinct free variables* $x_1, \ldots, x_m$ *of wff* B*,* and, where $\{A\}$*[x/t] is the result of simultaneously substituting for each free occurrence of* x *in the wff* A*, the term* t*.*

PA has a decidable language.

The first four axioms constitute the recursive definitions of $+$ and $\cdot$. The axioms of the form (22) all together constitute a statement of the principle of mathematical induction.

PA suffices to prove all the theorems in *elementary* number theory books! It has as its *standard normal model* the obvious one for the arithmetic of non-negative integers.[3] It *was* originally *intended* to be complete[4] (but, thanks to Gödel's First Incompleteness Theorem from 1931, we now know it is not — proof omitted).

PA is not decidable (proof omitted), but its theorems can be algorithmically listed.

(Problem A4 continues onto the next page.)

[3]A *normal* model is (by definition) one in which the interpretation of the equality *symbol* is equality on the model's universe/domain of discourse.

In PA's standard normal model, the universe/domain of discourse is the (countable) set of non-negative integers, each constant symbol n is interpreted as the non-negative integer n, $+$ is interpreted as addition of the non-negative integers, … .

[4]This is since it is and was supposed to be an (algorithmically decidable) axiomatization *of a fixed standard normal model.* By contrast the first order theory of groups (not detailed here) was intended to axiomatize a whole giant, interesting collection of rather disparate normal models.

(This page is the continuation of Problem A4.)

Definition T_{PA} *(called:* the f.o. theory of arithmetical truth*) has the same language as* PA *above and its axioms are the entire set of cwffs true in the standard normal model of* PA*.*

Trivially, T_{PA} is complete. However, it is not decidable and has no (algorithmically decidable) axiomatization (proofs omitted).

What is to be done for this problem: *Follow the hint just below to employ the Compactness Theorem as well as a suitable form of the Skolem-Löwenheim Theorem to prove* in*formally that* T_{PA} *also has a* countable non-*standard normal model, i.e., a countable one* not *isomorphic to its countable standard normal model.*

Hint: Form an extension $\mathrm{T}'_{\mathrm{PA}}$ of T_{PA} by adding a *new* constant symbol **i** and the *additional* infinite set of axioms

$$I = \{\mathbf{i} > 0, \mathbf{i} > 1, \mathbf{i} > 2, \ldots\}, \tag{23}$$

where, for terms t_1,t_2 of $\mathrm{T}'_{\mathrm{PA}}$,

$$t_1 > t_2 \tag{24}$$

is an abbreviation for the wff, also of $\mathrm{T}'_{\mathrm{PA}}$,

$$(\exists v_1)[t_1 = t_2 + (v_1 + 1)]. \tag{25}$$

Show informally that each *finite* subset of the axioms of $\mathrm{T}'_{\mathrm{PA}}$ has a normal model.

Explain how to conclude from Compactness that $\mathrm{T}'_{\mathrm{PA}}$ itself has a normal model.

Explain how to conclude from a suitable form of the Skolem-Löwenheim Theorem that $\mathrm{T}'_{\mathrm{PA}}$ has a *countable* normal model.

Show informally this new, countable normal model *restricted to the language of just* T_{PA} can*not* be isomorphic to the (countable) standard normal model of T_{PA}. To do this, show that, in the new normal model itself, the constant symbol **i** has a interpretation very unlike the *standard* meaning of any of the constant symbols *of* PA. N.B. The restricted version of the new normal model still has the same universe/domain of discourse; therefore, the object which is the *interpretation of* **i** from the unrestricted version *is still in there.*