C1. **Theory** (25 points)

a. (5 points)

Prove that $\{w \in \{a, b\}^* \mid$ the number of occurrences of ab in w equals the number of occurrences of ba in $w\}$ is a regular language.

b. (5 points)

Prove that $\{w \in \{a, b, c\}^* \mid$ the number of occurrences of ab in w equals the number of occurrences of ba in $w\}$ is *not* a regular language.

c. (7 points)

Prove that no infinite subset of $\{a^n b^n \mid n \geq 0\}$ is a regular language.

d. (8 points)

Prove that $\{a^n b^m \mid n \geq 0, m \geq 0, m \neq n\}$ is a context free language.

C2 **Theory** (25 points)

Let $A = \{a_1, \ldots, a_n\}$ where $n > 1$. Let

$$L = \{w \in A^* \mid w \text{ is missing at least one symbol of A}\}.$$

a. (5 points)

Explicitly exhibit the state diagram of a non-deterministic finite automaton, M, having exactly $n + 1$ states and such that $L(M) = L$.

b. (10 points)

Show that any spanning set of L must contain at least 2^n members. Recall the crucial equivalence relation used in the Myhill-Nerode theorem applied to L is $\equiv_L$, where for all $x, y \in A^*$,

$$x \equiv_L y \overset{\text{def}}{\Leftrightarrow} (\forall z \in A^*)[xz \in L \Leftrightarrow yz \in L]$$

Hint: Let $B, C \subseteq A$ where $B \neq C$. Let x contain all and only the symbols in B and likewise let y contain all and only the symbols in C. Show $x \not\equiv_L y$ (consider a symbol that is in one of the two sets (B or C) but not the other).

c. (10 points)

Let L_1 and L_2 be regular languages. Show (precisely) that the following set is also regular:

$$L = \{x \mid (x \in L_1 \ \&\ x \notin L_2) \text{ or } (x \notin L_1 \ \&\ x \in L_2)\}$$

C3 Theory (25 points)

Fix a standard programming formalism φ for computing all the *one-argument* partial computable functions which map the non-negative integers into themselves. Code (Gödel) number the φ-programs *onto* the entire set of non-negative integers. Let φ_p denote the partial function computed by program (number) p in the φ-system. Let $W_p \stackrel{\text{def}}{=}$ the domain of φ_p.[1] You may assume *without* proof that in the φ-system Universality, S-m-n, *and the Kleene Recursion Theorem (KRT)* etc. hold. You may also assume that there is a program that decides the predicate whether a program with code number p halts on x within t steps.

a. (5 points)

Prove that $\{x \mid W_x \text{ is empty }\}$ is not recursive.

b. (10 points)

Let $P(x) \stackrel{\text{def}}{\Leftrightarrow} [\varphi_x(x)\downarrow \;\&\; \varphi_x(x) \neq x]$. Show that this predicate is not computable.

c. (10 points)

Show that there is a non-negative number p such that $\forall x[\varphi_p(x)\downarrow \text{ iff } x = p]$, i.e., $W_p = \{p\}$.

[1] Then $W_0, W_1, W_2, \ldots$ provides a standard listing of *all* the re sets (of non-negative integers).

C4 Theory (25 points)

Fix a standard programming formalism φ for computing all the *one-argument* partial computable functions which map the non-negative integers into themselves. Code (Gödel) number the φ-programs *onto* the entire set of non-negative integers. Let φ_p denote the partial function computed by program (number) p in the φ-system. Let $W_p \stackrel{\text{def}}{=}$ the domain of φ_p.[1] You may assume *without* proof that in the φ-system Universality, S-m-n, *and the Kleene Recursion Theorem (KRT)* hold. You may also assume that there is a program that decides the predicate whether a program with code number p halts on x within t steps.

The first two parts of this question will lead you (with *very* useful hints) through a proof of the following

Theorem 1 *Suppose* Δ is a collection of re sets. *Let*

$$P_\Delta \stackrel{\text{def}}{=} \{p \mid W_p \in \Delta\}. \tag{1}$$

Suppose P_Δ is r.e.
Then

$$(\forall p)[W_p \in \Delta \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta]]. \tag{2}$$

a. (8 points)
Assume all the hypotheses of the Theorem. *Explicitly use KRT* in the φ-system (formally or informally — as you choose) to prove that

$$(\forall p)[W_p \in \Delta \Rightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta]]. \tag{3}$$

Hint for C4(a): Suppose that $W_p \in \Delta$. Suppose for contradiction that ($\forall$ finite sets $D \subseteq W_p)[D \notin \Delta]$. Apply KRT to obtain an self-referential e which determines its I/O behavior on input x *in part* according to whether or not "e appears in P_Δ within x steps." Make this precise, figure out what to have e do in each case, etc., and get a contradiction.

b. (8 points)
Assume all the hypotheses of the Theorem. *Explicitly use KRT* in the φ-system (formally or informally — as you choose) to prove that

$$(\forall p)[(\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta] \Rightarrow W_p \in \Delta]. \tag{4}$$

Hint for C4(b): Suppose ($\exists$ a finite set $D \subseteq W_p)[D \in \Delta]$. Let D be an example. Suppose for contradiction that $W_p \notin \Delta$. Apply KRT to obtain an self-referential e which determines its I/O behavior on input x *in part* according to whether it eventually discovers that "$[x \in D \vee e$ appears in $P_\Delta]$." Make this precise, figure out what to have e do if it makes this discovery, *etc.*

c. (9 points)
Let

$$A = \{p \mid W_p = \{0\}\}. \tag{5}$$

Explicitly use the Theorem stated above in this question, C4, to show that A is *not* r.e., *where* A is defined in (5) above.

Hint for C4(c): Suppose for contradiction otherwise. Clearly $A = P_\Delta$ for $\Delta = \{\{0\}\}$. Therefore, from (2) above, we have that $(\forall p)[W_p = \{0\} \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D = \{0\}]]$. Pick D and W_p so that $D = \{0\} \subseteq W_p \neq \{0\}$. Get a contradiction.

[1] Then $W_0, W_1, W_2, \ldots$ provides a standard listing of *all* the re sets (of non-negative integers).