Outline

- Introduction
- Lossless compression
- Lossy compression
  - metrics
  - general methods
    - scalar
    - vector
    - differential
    - transform
      - Haar   Tutorial version
      - JPEG
      - wavelet
  - MPEG

Multiplication problem

XII
* IX
CVIII

transform

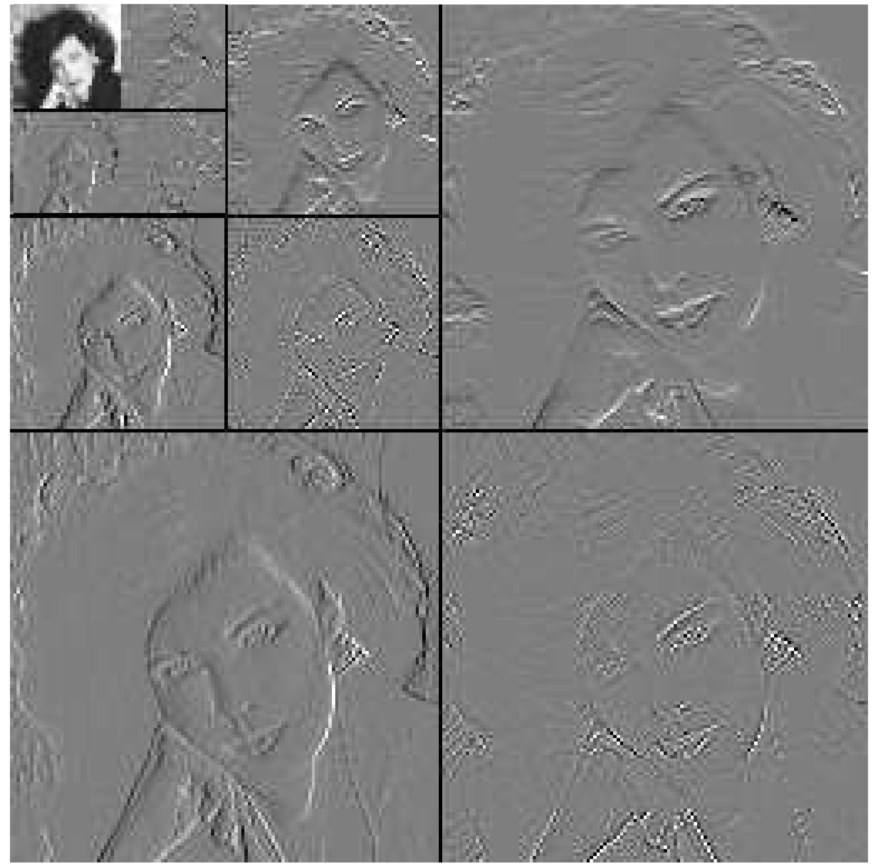
inverse transform

12
* 9
108

useful application

Our useful application is QUANTIZATION and COMPRESSION!

General Methods - Video Subband Encoding
HAAR Wavelets



HAAR Wavelets
" averages and differences "

original 'image' [8 4 1 3] [a b c d]

transformed 'image' (level 1) [6 2 2 -1] $\left[\frac{a+b}{2} \quad \frac{c+d}{2} \quad \frac{a-b}{2} \quad \frac{c-d}{2} \right]$

averages (6 2) details (2 -1)

transformed 'image' (level 2) [4 2 2 -1] $\left[\frac{a+b+c+d}{4} \quad \frac{a+b-c-d}{4} \quad \frac{a-b}{2} \quad \frac{c-d}{2} \right]$

average of averages (4) detail (2)

1-Dimension 2-level Haar Transform (N=4)

1st level basis matrices N=4

Step 1 – 1st level

$$[\,8\ \ 4\ \ 1\ \ 3\,] \bullet \begin{bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \end{bmatrix}$$

$$= [\,6\ \ \ 2\ \ \ 2\ \ \ -1\,]$$

2nd level basis matrices N=4

Step 2 – 2nd level

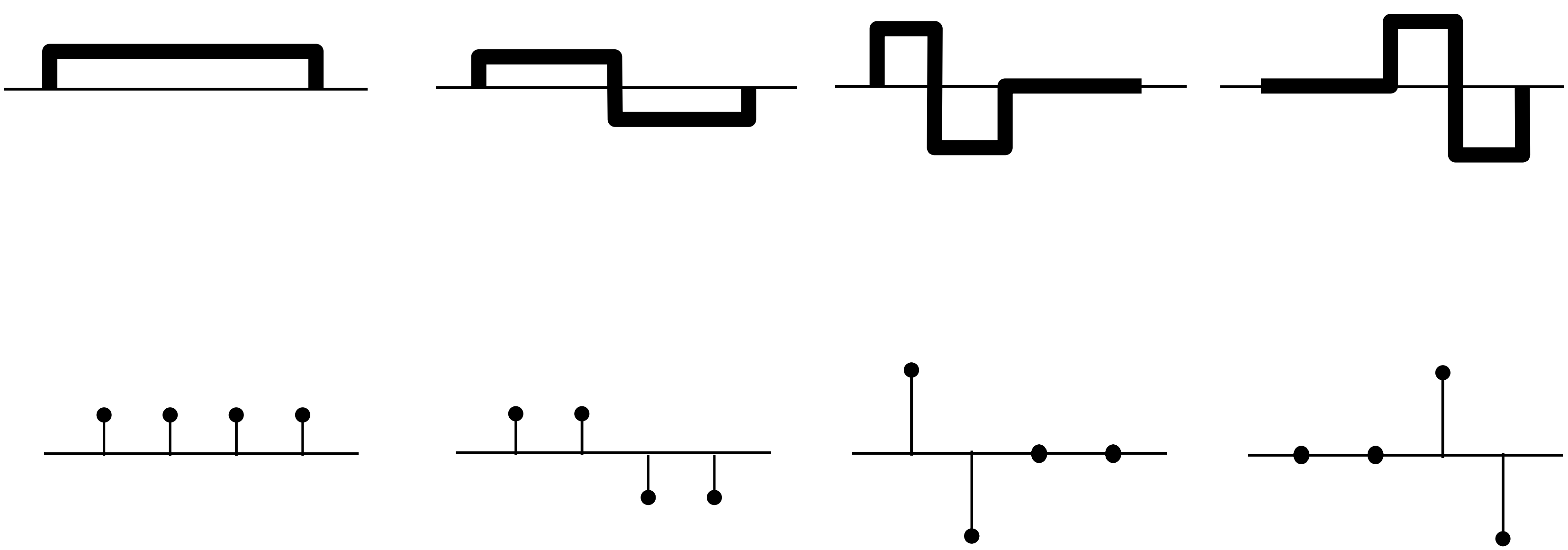
$$[\,6\ \ 2\ \ 2\ \ -1\,] \bullet \begin{bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{bmatrix}$$

$$= [\,4\ \ \ 2\ \ \ 2\ \ \ -1\,]$$

2-level Haar Basis Matrices N=4

$$[\,8 \quad 4 \quad 1 \quad 3\,] \bullet \left(\begin{bmatrix} \tfrac{1}{4} & \tfrac{1}{4} & \tfrac{1}{4} & \tfrac{1}{4} \end{bmatrix} \; \begin{bmatrix} \tfrac{1}{4} & \tfrac{1}{4} & -\tfrac{1}{4} & -\tfrac{1}{4} \end{bmatrix} \; \begin{bmatrix} \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 \end{bmatrix} \; \begin{bmatrix} 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} \end{bmatrix} \right)$$

$$= [\,4 \quad 2 \quad 2 \quad -1\,]$$



Deriving 2-level Haar Basis Matrices N=4

1st level basis matrices X **2nd level basis matrices**

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

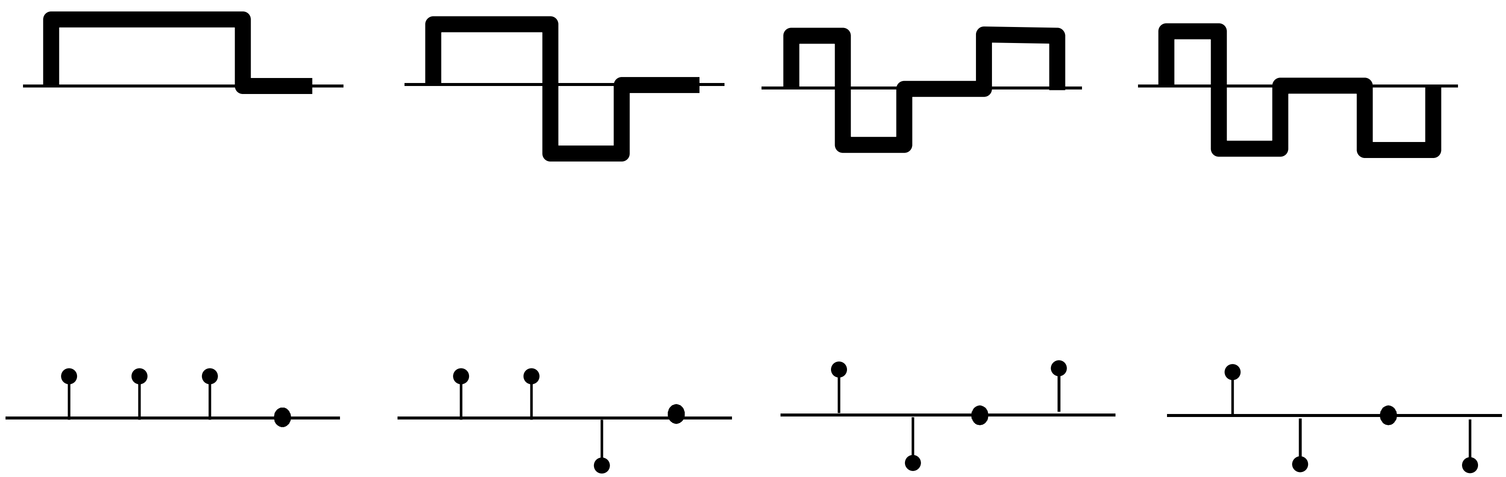
2-level basis matrices

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}$$

2-level Inverse Haar Basis Matrices N=4

$$[\,4\ \ 2\ \ 2\ \ -1\,] \bullet \begin{bmatrix} \begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix} & \begin{pmatrix}1 & 1 & -1 & 0\end{pmatrix} & \begin{pmatrix}1 & -1 & 0 & 1\end{pmatrix} & \begin{pmatrix}1 & -1 & 0 & -1\end{pmatrix} \end{bmatrix}$$

$$= [\,8\ \ 4\ \ 1\ \ 3\,]$$



- Note: process is reversible without loss

Forward transform • Inverse transform = ??

$$[\,8 \quad 4 \quad 1 \quad 3\,] \bullet \left(\text{transform}\right) \bullet \left(\text{transform}^{-1}\right) = [\,8 \quad 4 \quad 1 \quad 3\,]$$

$$\begin{pmatrix} ¼ & ¼ & ½ & 0 \\ ¼ & ¼ & -½ & 0 \\ ¼ & -¼ & 0 & ½ \\ ¼ & -¼ & 0 & -½ \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Compression - Quantization

[4   2   2 -1]

quantize ⇓   25% compression ??

$$[\,4 \;\; 2 \;\; 2 \;\; 0\,] \bullet \left(\begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix} \; \begin{bmatrix}1 & 1 & -1 & 0\end{bmatrix} \; \begin{bmatrix}1 & -1 & 0 & 1\end{bmatrix} \; \begin{bmatrix}1 & -1 & 0 & -1\end{bmatrix} \right) = [\,8 \;\; 4 \;\; 2 \;\; 2\,]$$

What is SNR ?

original [8   4   1   3]

decoded [8   4   2   2]

$$\frac{(8^2 + 4^2 + 1^2 + 3^2)/4}{((8-8)^2 + (4-4)^2 + (1-2)^2 + (3-2)^2)/4} = \frac{90}{2} = 45$$

What would SNR be if we quantized to [4 2 0 -1] ?

Intuition: it's better to quantize values closer to 0.

1-Dimension 1-level Haar Transform

before

after

200	200	200	202	…
196	200	202	202	…
196	198	200	200	…
194	194	196	198	…
…				

200	201 …	0	-1	…
198	202 …	-2	0	…
197	200 …	-1	0	…
194	197 …	0	-1	…
…				

1-Dimension 3-level Haar Transform (N=8)

	12	8	10	6	-4	-2	8	-4
Step 1 – 1st level	10	8	-3	2	2	2	-1	6
Step 2 – 2nd level	9	-½	1	-2½	2	2	-1	6
Step 3 – 3rd level	4¼	4 ¾	1	-2½	2	2	-1	6

1-Dimension 3-level Haar Transform Basis Matrices (N=8)

row 0 $\quad \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

row 1 $\quad \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$

row 2 $\quad \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$

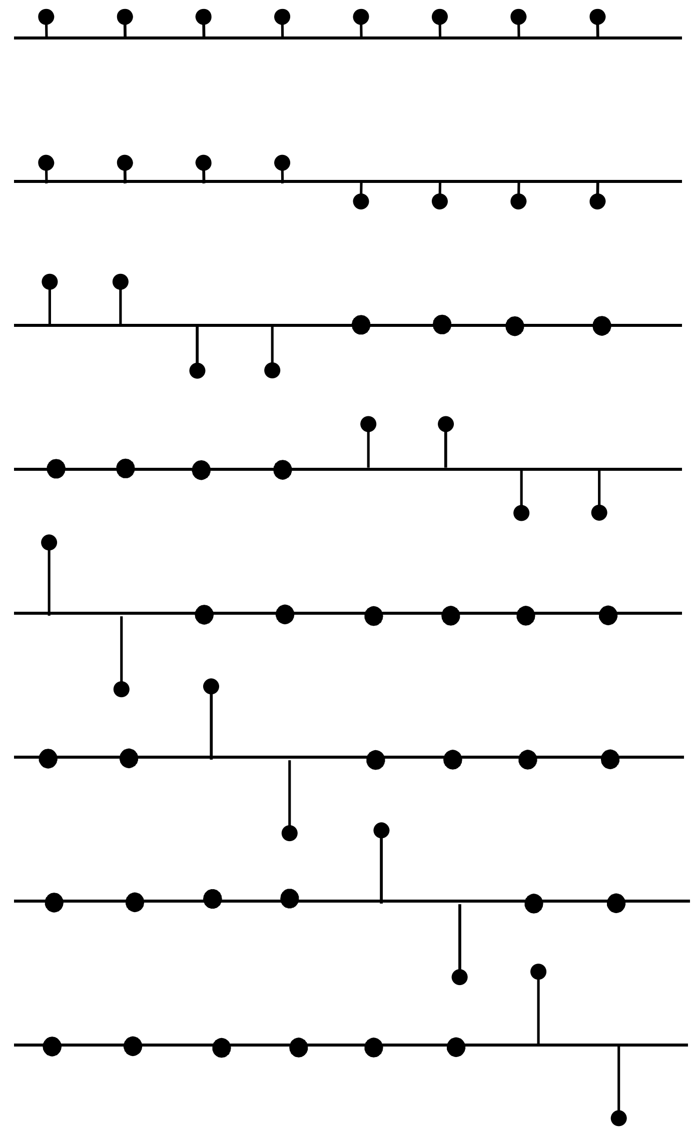
row 3 $\quad \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix}$

row 4 $\quad \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

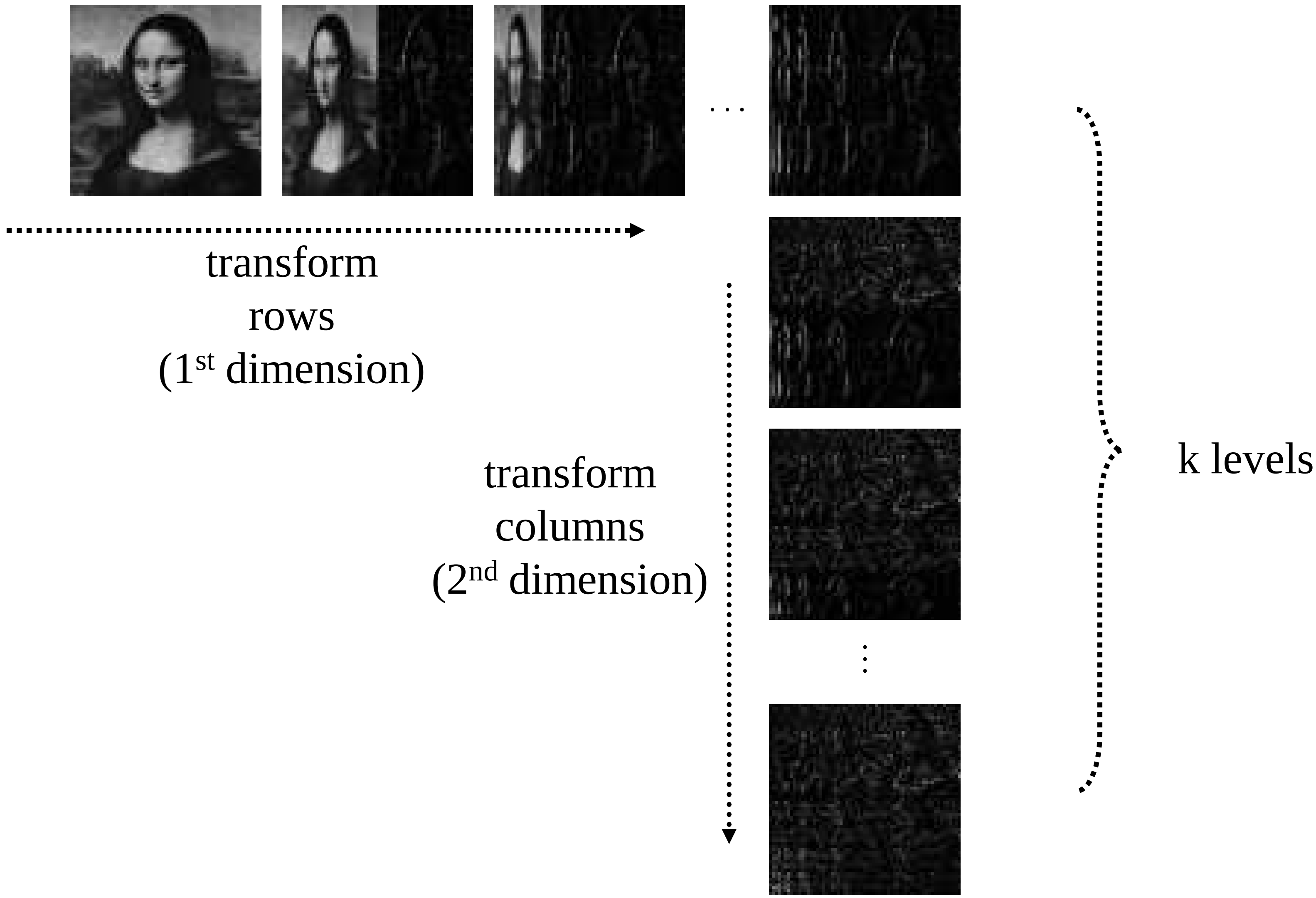
row 5 $\quad \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$

row 6 $\quad \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$

row 7 $\quad \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$



2-Dimension Multilevel Haar Transform



Exercise: Perform a 2-D 2-level Haar standard transform on the following 4 X 4 pixel image

18	8	9	1
20	6	11	15
16	8	6	6
6	6	-2	2

→

13	5	5	4
13	13	7	-2
12	6	4	0
6	0	0	-2

→

9	4	5	4
13	0	7	-2
9	3	4	0
3	3	0	-2

↓

11	2	6	1
6	3	2	-1
-2	2	-1	3
3	0	2	1

↓

8.5	2.5	4	0
2.5	-.5	2	1
-2	2	-1	3
3	0	2	1

Exercise: 2-D 2-level Haar standard transform on 4 X 4 pixel image

$$
\begin{matrix} 18 & 8 & 9 & 1 \\ 20 & 6 & 11 & 15 \\ 16 & 8 & 6 & 6 \\ 6 & 6 & -2 & 2 \end{matrix}
\bullet
\begin{pmatrix}
(\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac12\ -\tfrac12\ 0\ 0) & (0\ 0\ \tfrac12\ -\tfrac12) \\
(\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac12\ -\tfrac12\ 0\ 0) & (0\ 0\ \tfrac12\ -\tfrac12) \\
(\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac12\ -\tfrac12\ 0\ 0) & (0\ 0\ \tfrac12\ -\tfrac12) \\
(\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac12\ -\tfrac12\ 0\ 0) & (0\ 0\ \tfrac12\ -\tfrac12)
\end{pmatrix}
\Rightarrow
\begin{matrix} 9 & 4 & 5 & 4 \\ 13 & 0 & 7 & -2 \\ 9 & 3 & 4 & 0 \\ 3 & 3 & 0 & -2 \end{matrix}
$$

transform EACH row
with 1-D 2-level basis matrix

$$
\bullet
\begin{pmatrix}
(\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) & (\tfrac14\ \tfrac14\ \tfrac14\ \tfrac14) \\
(\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) & (\tfrac14\ \tfrac14\ -\tfrac14\ -\tfrac14) \\
(\tfrac12\ -\tfrac12\ 0\ 0) & (\tfrac12\ -\tfrac12\ 0\ 0) & (\tfrac12\ -\tfrac12\ 0\ 0) & (\tfrac12\ -\tfrac12\ 0\ 0) \\
(0\ 0\ \tfrac12\ -\tfrac12) & (0\ 0\ \tfrac12\ -\tfrac12) & (0\ 0\ \tfrac12\ -\tfrac12) & (0\ 0\ \tfrac12\ -\tfrac12)
\end{pmatrix}
$$

transform EACH column
with rotated 1-D 2-level
basis matrix

$$
\Downarrow
$$

8.5	2.5	4	0
2.5	-.5	2	1
-2	2	-1	3
3	0	2	1

Exercise: Derive the 2-Dimension, 2-level Haar standard basis matrices for N=4

$$
\begin{matrix} 18 & 8 & 9 & 1 \\ 20 & 6 & 11 & 15 \\ 16 & 8 & 6 & 6 \\ 6 & 6 & -2 & 2 \end{matrix}
\bullet
\left(
\begin{matrix}
\frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} &
\frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} &
\frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \\
\frac{1}{16}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} &
\frac{1}{16}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} &
\frac{1}{8}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \\
\frac{1}{8}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} &
\frac{1}{8}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} &
\frac{1}{4}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \\
\frac{1}{8}\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} &
\frac{1}{8}\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} &
\frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}
\end{matrix}
\right)
$$

$$
= \begin{matrix} 8.5 & 2.5 & 4 & 0 \\ 2.5 & -.5 & 2 & 1 \\ -2 & 2 & -1 & 3 \\ 3 & 0 & 2 & 1 \end{matrix}
$$

Compression

8.5	2.5	4	0
2.5	-.5	2	1
-2	2	-1	3
3	0	2	1

quantize ≤ |1.0|

8.5	2.5	4	0
2.5	0	2	0
-2	2	0	3
3	0	2	0

compression
6 out of 16

inverse

reconstruction

19.5	7.5	7.5	1.5
19.5	7.5	9.5	15.5
15.5	7.5	6.5	6.5
5.5	5.5	0.5	0.5

original

18	8	9	1
20	6	11	15
16	8	6	6
6	6	-2	2

What is SNR ?

$$\frac{(18^2 + 8^2 + 9^2 + \ldots + 2^2) / 16}{((18-19.5)^2 + (8-7.5)^2 + (9-7.5)^2 + \ldots + (2-0.5)^2) / 16}$$

$= 6.703 = 8.3\text{dbs}$

Exercise: Derive the 2-Dimension, 2-level Haar standard basis matrices for N=4

$$
\begin{pmatrix} 18 & 8 & 9 & 1 \\ 20 & 6 & 11 & 15 \\ 16 & 8 & 6 & 6 \\ 6 & 6 & -2 & 2 \end{pmatrix}
\bullet
\begin{pmatrix}
\frac{1}{16}\begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} &
\frac{1}{16}\begin{bmatrix} 1&1&1&1\\1&1&1&1\\-1&-1&-1&-1\\-1&-1&-1&-1 \end{bmatrix} &
\frac{1}{8}\begin{bmatrix} 1&1&1&1\\-1&-1&-1&-1\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\frac{1}{8}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&1&1&1\\-1&-1&-1&-1 \end{bmatrix} \\
\frac{1}{16}\begin{bmatrix} 1&1&-1&-1\\1&1&-1&-1\\1&1&-1&-1\\1&1&-1&-1 \end{bmatrix} &
\frac{1}{16}\begin{bmatrix} 1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1 \end{bmatrix} &
\frac{1}{8}\begin{bmatrix} 1&1&-1&-1\\-1&-1&1&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\frac{1}{8}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&1&-1&-1\\-1&-1&1&1 \end{bmatrix} \\
\frac{1}{8}\begin{bmatrix} 1&-1&0&0\\1&-1&0&0\\1&-1&0&0\\1&-1&0&0 \end{bmatrix} &
\frac{1}{8}\begin{bmatrix} 1&-1&0&0\\1&-1&0&0\\-1&1&0&0\\-1&1&0&0 \end{bmatrix} &
\frac{1}{4}\begin{bmatrix} 1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\frac{1}{4}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&-1&0&0\\-1&1&0&0 \end{bmatrix} \\
\frac{1}{8}\begin{bmatrix} 0&0&1&-1\\0&0&1&-1\\0&0&1&-1\\0&0&1&-1 \end{bmatrix} &
\frac{1}{8}\begin{bmatrix} 0&0&1&-1\\0&0&1&-1\\0&0&-1&1\\0&0&-1&1 \end{bmatrix} &
\frac{1}{4}\begin{bmatrix} 0&0&1&-1\\0&0&-1&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\frac{1}{4}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&-1\\0&0&-1&1 \end{bmatrix}
\end{pmatrix}
$$

Basis matrix (1,2)

$$
= \begin{matrix} 8.5 & 2.5 & 4 & 0 \\ 2.5 & -.5 & 2 & 1 \\ -2 & 2 & -1 & 3 \\ 3 & 0 & 2 & 1 \end{matrix}
$$

Exercise: Derive the 2-Dimension, 2-level Haar basis

Consider 2-Dimension 2-level basis matrix (1,2)

$$1/8 \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/8 & 1/8 & -1/8 & -1/8 \\ -1/8 & -1/8 & 1/8 & 1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1-Dimension 2-level basis matrices

row 0	¼	¼	¼	¼
row 1	¼	¼	-¼	-¼
row 2	½	-½	0	0
row 3	0	0	½	-½

take cross product row 1 by row 2

X	¼	¼	-¼	-¼
½	1/8	1/8	-1/8	-1/8
-½	-1/8	-1/8	1/8	1/8
0	0	0	0	0
0	0	0	0	0

Haar Transform

original (256 X 256)

(a±b)/2

normalized

$(a \pm b)/(2^{1/2})$

compression 10 : 1

Normalized Haar Wavelet

using $(a \pm b)/(2^{½})$

original

25 : 1

100 : 1

Note: Images of Emmy Noether thanks to Prof Mulcahy (Spelman College)