

Some Results on U-shaped, Vacillatory and Team Learning

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Abstract. *U-shaped learning behaviour* in cognitive development involves learning, unlearning and relearning. It occurs, for example, in learning irregular verbs. The prior cognitive science literature is occupied with how humans do it, for example, general rules versus tables of exceptions. This paper is mostly concerned with whether U-shaped learning behaviour may be *necessary* in the abstract mathematical setting of inductive inference, that is, in the computational learning theory following the framework of Gold. All notions considered are learning from text, that is, from positive data. Previous work showed that U-shaped learning behaviour is necessary for behaviourally correct learning but not for syntactically convergent, learning in the limit (= explanatory learning). The present paper establishes the necessity for the whole hierarchy of classes of vacillatory learning where a behaviourally correct learner has to satisfy the additional constraint that it vacillates in the limit between at most k grammars, where $k \geq 1$. Non U-shaped vacillatory learning is shown to be restrictive: Every non U-shaped vacillatorily learnable class is already learnable in the limit. Furthermore, if vacillatory learning with the parameter $k = 2$ is possible then non U-shaped behaviourally correct learning is also possible. But for $k = 3$, surprisingly, there is a class witnessing that this implication fails.

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1 Introduction and Motivation

U-shaped learning is a learning behaviour in which the learner first learns the correct behaviour, then abandons the correct behaviour and finally returns to the correct behaviour once again. This pattern of learning behaviour has been observed by cognitive and developmental psychologists in a variety of child development phenomena, such as language learning [7, 19, 29], understanding of temperature [29, 30], understanding of weight conservation [6, 29], object permanence [6, 29] and face recognition [8]. The case of language acquisition is paradigmatic. In the case of the past tense of English verbs, it has been observed that children learn correct syntactic forms (call/called, go/went), then undergo a period of overregularization in which they attach regular verb endings such as ‘ed’ to the present tense forms even in the case of irregular verbs (break/broke, speak/spoke) and finally reach a final phase in which they correctly handle both regular and irregular verbs. We note that, if U-shaped learning is seen for such natural subparts [9, Section 4] of a whole language, then, because the phenomenon already appears in subparts, it therefore appears in learning the whole language. The irregular verb examples of U-shaped learning behaviour has figured so prominently in the so-called “Past Tense Debate” in cognitive science that competing models of human learning are often judged on their capacity for modeling the U-shaped learning phenomenon [19, 25, 31].

The prior literature is typically concerned with modeling *how* humans achieve U-shaped behaviour. Recently, Baliga, Case, Merkle, Stephan and Wiehagen [2] looked at *why* humans exhibit this seemingly inefficient behaviour. Is it a mere harmless evolutionary accident or is it *necessary* for full human learning power? Specifically, are there some learning tasks for which U-shaped behaviour is logically necessary? In the present paper we continue this line of work.

In order to explain our results, we have to be a bit more formal and refer to Section 2 below for an explanation of the mathematical terms used. Baliga, Case, Merkle, Stephan and Wiehagen [2] initiated the Gold style learning theoretic study of U-shaped learning behaviour and showed that it is circumventable for **TextEx**-learning, see Theorem 5. In contrast to this, Fulk, Jain and Osherson’s proof of [16, Theorem 4] shows that U-shaped learning behaviour is necessary for the full learning power of **TextBc**-learning. We show in Theorem 7 below that U-shaped learning behaviour is also necessary for full learning power for the whole hierarchy of the learning criteria **TextFex_b** *strictly* between **TextEx** and **TextBc**. While Case [9] proved that the **TextFex_b** criteria form a hierarchy of more and more powerful learning criteria, Theorem 7 of the present paper shows that *non* U-shaped **TextFex_b**-learners are not more powerful than **TextEx**-learners and the hierarchy collapses to **TextFex₁** if U-shaped behaviour is forbidden. In other words, there are classes of languages that can be **TextFex_n**-identified, for $n > 1$, but these learners *must* be U-shaped on some texts. What if we consider the more liberal criterion **TextBc**? Our Theorem 18 strengthens the collapse result of Theorem 7 considerably by showing that there are classes in **TextFex₃** (and therefore in **TextFex_n** for all $n > 3$) that cannot be **TextBc**-learned by a non U-shaped learner. This means that U-shaped learning behaviour cannot be dispensed with for learning such classes, even if we only require behavioural convergence and permit convergence to possibly infinitely many syntactically different correct hypotheses. By contrast, our last main result, Theorem 17, shows that every class of languages that can be **TextFex₂**-identified can be **TextBc**-identified by a non U-shaped learner. Hence, for only this early stage of the hierarchy, the cases in which **TextFex₂**-identification necessitates U-shaped learning behaviour can be circumvented by shifting to **TextBc**-identification. We note that the relevance of Gold style learning to cognitive science has been supported in the cognitive science

literature, for *example*, in [18, 24]. The publications [9, 32] critically discuss the relevance of the well studied criterion **TxtBc** to human learning; in order to avoid overgeneralization and mere interpolation of the data, one might want that a learner does not generate larger and larger hypotheses but tries to concentrate on finding few correct ones. When **TxtEx**-learning is impossible, this can only be done by vacillating between some few hypotheses. Case [9] formalized this approach by introducing the criteria **TxtFex_b** for small, feasibly sized $b > 1$. Case also argues that these criteria *may* better fit the human case than the **TxtEx** criterion. Certainly, then, the new results of the present paper, regarding, for example, the **TxtFex₃** criterion are of interest for cognitive science, and *may* inform regarding the human case.

A further interesting aspect is that this papers gives a close relation between vacillatory learning and team learning. Theorem 2 gives the basic connection: A class is **TxtFex_b**-learnable iff there is a team of b learners where all team-members converge on every text of a language to be learned and at least one of the team-members has to be correct. Furthermore, in Sections 3 and 6 some general inclusions for non U-shaped team learning are established.

In Section 2 we introduce the notation and the basic definitions for the rest of the paper. We also include the basic results from [2] which are relevant for the present paper. In Section 3 we show that **NUShTxFex_b**, actually coincides with **TxtEx** so that on one hand every **TxtEx**-learner can be replaced by a non U-shaped one while on the other hand vacillatory non U-shaped learners do not have more power than non U-shaped **TxtEx**-learners; that is, **NUShTxFex_b** = **TxtEx** for all $b \in \{1, 2, 3, \dots, *\}$. In the subsequent sections we investigate the question whether one can obtain a non U-shaped learner from a **TxtFex_b**-learner if one is willing to give up some constraints on the number of hypotheses considered. The positive result **TxtFex₂** \subseteq **NUShTxBc** is presented in Section 4 and the negative result **TxtFex₃** $\not\subseteq$ **NUShTxBc** is presented in Section 5. Section 6, as well as some previous sections, provides some results on non U-shaped team learning.

2 Preliminaries

\mathbb{N} denotes the set of natural numbers, $\{0, 1, 2, \dots\}$. Unless otherwise specified the variables a, c, d, e, i, j, k range over \mathbb{N} . D, P, S range over *finite* sets of natural numbers. The cardinality function is denoted by $\text{card}(\cdot)$. $\text{card}(D) \leq *$ means that $\text{card}(D)$ is finite. The symbol $*$ is used to denote the “finite with no preassigned bound”. Unless otherwise stated, b will range over $\mathbb{N} \cup \{*\}$. The symbols $\subseteq, \subset, \supseteq, \supset$ respectively denote the subset, proper subset, superset and proper superset relation between sets. The union \cup and the intersection \cap are defined as usual; \in denotes set-theoretic membership and $-$ denotes set-theoretic difference. The quantifiers \forall^∞ and \exists^∞ mean “for all but finitely many” and “there exists infinitely many”, respectively.

A pair $\langle i, j \rangle$ stands for an arbitrary, computable one-to-one encoding of all pairs of natural numbers onto \mathbb{N} [26]. Similarly we can define $\langle \cdot, \dots, \cdot \rangle$ for encoding n -tuples of natural numbers, for $n > 1$, onto \mathbb{N} .

φ denotes a fixed *acceptable* programming system for the partial computable functions [26]. φ_e denotes the partial computable function computed by the program with code number e in the φ -system. We will unambiguously refer to programs using their code number in the φ -system. We let H, I, J, L range over recursively enumerable sets and \mathcal{L} range over classes of recursively enumerable sets. K denotes the diagonal halting problem which is the standard example for a nonrecursive r.e. set.

We fix $W_{e,s}$ to denote W_e enumerated within s computation steps [5]. For our purposes, we need a more specialized version of complexity which satisfies additionally the following constraints which can be easily ensured using standard techniques:

- W_e is the domain of φ_e and $W_e = \bigcup_s W_{e,s}$;
- $W_{e,0} = \emptyset$ and $W_{e,s} \subseteq W_{e,s+1} \subseteq \{0, \dots, s\}$ for all e, s ;
- $\{(e, s, x) : x \in W_{e,s}\}$ is recursive;
- $\{(x, s) : x \in W_{e,s}\}$ is primitive recursive for all e ;
- For every primitive-recursive enumeration A_s of some set A with $A_0 = \emptyset \wedge (\forall s) [A_s \subseteq A_{s+1} \subseteq \{0, \dots, s\}]$ there is an index e with $(\forall s) [W_{e,s} = A_s]$; furthermore, e can be computed from an index of the enumeration for A_s .

Any unexplained recursion-theoretic notions are from [20, 26].

We now introduce the basic definitions of Gold-style computational learning theory.

A *sequence* σ is a mapping from an initial segment of \mathbb{N} into $\mathbb{N} \cup \{\#\}$. An *infinite sequence* is a mapping from \mathbb{N} into $\mathbb{N} \cup \{\#\}$. The content of a finite or infinite sequence σ is the set of natural numbers occurring in σ and is denoted by $\text{content}(\sigma)$. The length of a sequence σ is the number of elements in the domain of σ and is denoted by $|\sigma|$. For a subset L of \mathbb{N} , $\text{seg}(L)$ denotes the set of sequences σ with $\text{content}(\sigma) \subseteq L$. An infinite sequence T is a *text for* L iff $L = \text{content}(T)$.

The symbol $\#$ is mainly introduced to uniformly deal with the empty language and intuitively represents a pause in the presentation of the language to the learner. Concatenation of two sequences σ and τ is denoted by $\sigma\tau$. If $x \in (\mathbb{N} \cup \{\#\})$, then σx means $\sigma\tau$ where τ is the sequence consisting of exactly one element which is x . $\sigma \subseteq \tau$ means that σ is an initial segment of τ and $\sigma \subset \tau$ means that σ is a proper initial segment of τ .

Intuitively, a *text* for a language L is an infinite stream or sequential presentation of *all* the elements of the language L in any order and with the $\#$'s representing pauses in the presentation of the data. For example, the only text for the empty language is an infinite sequence of $\#$'s. Technically, a text is a mapping from \mathbb{N} into $(\mathbb{N} \cup \{\#\})$. We let T , with possible subscripts and superscripts, range over texts. $T[n]$ denotes the finite initial segment of T with length n . Observe that the domain of $T[n]$ is $\{x : x < n\}$. So $T(n)$ is not a member of the sequence $T[n]$. $\sigma \subset T$ denotes the fact that σ is an initial segment of T . Observe that in this case we have $\sigma = T[|\sigma|]$.

A learner will map sequences to hypotheses. These are represented by natural numbers and interpreted as codes for programs in the φ -system. \mathbf{M} , with possible superscripts and subscripts, is intended to range over language learning machines.

Definition 1. [2, 3, 9, 12, 13, 17, 23] A *language learning machine* \mathbf{M} is a computable mapping from $\text{seg}(\mathbb{N})$ into \mathbb{N} . \mathbf{M} **TxtBc**-learns a class \mathcal{L} of r.e. languages iff for every $L \in \mathcal{L}$ and every text T for L , almost all hypotheses $\mathbf{M}(T[n])$ are indices for the language L to be learned.

A **TxtBc**-learner \mathbf{M} for \mathcal{L} is a **TxtFex_{*}**-learner iff for every $L \in \mathcal{L}$ and every text T for L the set $\{\mathbf{M}(T[n]) : n \in \mathbb{N}\}$ is finite.

A **TxtFex_{*}**-learner \mathbf{M} for \mathcal{L} is a **TxtFex_b**-learner for a $b \in \{1, 2, \dots\}$ iff there are for every $L \in \mathcal{L}$ and every text T for L at most b indices which \mathbf{M} outputs infinitely often, that is, $|\{e : (\exists^\infty n) [e = \mathbf{M}(T[n])]\}| \leq b$.

A **TxtBc**-learner \mathbf{M} for \mathcal{L} is a **TxtEx**-learner iff for every $L \in \mathcal{L}$ and every text T for L almost all hypotheses $\mathbf{M}(T[n])$ are the same grammar for L .

A **TxtBc**-learner \mathbf{M} for \mathcal{L} is non U-shaped iff for every $L \in \mathcal{L}$ and every text T for L there are no three numbers k, m, n such that $k < m < n$ and $W_{\mathbf{M}(T[k])} = L$, $W_{\mathbf{M}(T[m])} \neq L$ and $W_{\mathbf{M}(T[n])} =$

L . Furthermore, **NUShTxtBc**-learners, **NUShTxtFex_b**-learners and **NUShTxtEx**-learners (of a class \mathcal{L}) are those learners which are non U-shaped and at the same time a **TxtBc**-learner, **TxtFex_b**-learner and **TxtEx**-learner, respectively, (of \mathcal{L}).

The criteria **TxtBc**, **TxtFex_b**, **TxtEx**, **NUShTxtBc**, **NUShTxtFex_b**, **NUShTxtEx** are the sets consisting of all those classes which are learnable by a learner satisfying the respective above defined requirements.

The historically most important learning criterion is **TxtEx** where the learner has to converge syntactically to a single index of the language to be learned [17]. **TxtEx** stands for “explanatory identification from text”. Intuitively, the notion **TxtBc** captures what could be called learning in the most general sense, where “Bc” stands for “behaviourally correct” identification. In this, the learner outputs correct grammars almost always.

Intuitively a class \mathcal{L} of r.e. languages is **TxtFex_b** identified by a machine \mathbf{M} iff when \mathbf{M} is given as input *any* listing T of *any* $L \in \mathcal{L}$, it outputs a sequence of grammars such that, past some point in this sequence, no more than b *syntactically* different grammars occur and each of them is a grammar for L . The acronym **TxtFex** stands for ‘finite explanatory identification from text’. **TxtFex₁** is equivalent to **TxtEx**. Osherson and Weinstein [22] first studied the case with $b = *$, later Case [9] studied the whole hierarchy with $b \in \mathbb{N}^+$.

We say σ is a **TxtEx-stabilizing-sequence** [15] for a learner \mathbf{M} on a set L iff $\sigma \in \text{seg}(L)$ and $\mathbf{M}(\sigma\tau) = \mathbf{M}(\sigma)$ for all $\tau \in \text{seg}(L)$. Furthermore, σ is called a **TxtEx-locking-sequence** [4] for \mathbf{M} on L iff $W_{\mathbf{M}(\sigma)} = L$. Note that stabilizing and locking sequence definitions can be generalized to other learning criteria such as **TxtFex** and **TxtBc**; we often drop “**TxtEx**” (respectively, “**TxtBc**”, “**TxtFex**”) from “**TxtEx-stabilizing-sequence**” and “**TxtEx-locking-sequence**”, when it is clear from context.

Smith [27] studied learning by teams of machines, and, we can show vacillatory learning can be characterized by teams as follows.

Theorem 2. *A class \mathcal{L} has a **TxtFex_b**-learner iff there is a team of b machines $\mathbf{N}_1, \dots, \mathbf{N}_b$ such that for every $L \in \mathcal{L}$ and for every text T for L , each machine \mathbf{N}_a converges to a single index e_a and at least one of these indices e_a is an index for L .*

Case [9] showed **TxtFex₁** \subset **TxtFex₂** $\subset \dots \subset$ **TxtFex_{*}**, as stated in the following Theorem.

Theorem 3. [9] *For each $b \in \{1, 2, 3, \dots\}$, the separation **TxtFex_{b+1}** $\not\subseteq$ **TxtFex_b** is witnessed by the self-referential class $\mathcal{H}_b = \{W_e : e \in W_e \wedge |W_e \cap \{0, \dots, e\}| \leq b + 1\}$ consisting of all nonempty sets which have an index among its smallest $b + 1$ elements. Furthermore consider the class $\mathcal{H}_* = \{W_e : W_e \neq \emptyset \wedge e \leq \min(W_e)\}$ consisting of all nonempty sets where the smallest index is bounded by their smallest element. This class witnesses that **TxtFex_{*}** $\not\subseteq$ **TxtFex_b** for any $b \in \{1, 2, 3, \dots\}$.*

Proposition 4. **NUShTxtBc** $\not\subseteq$ **TxtFex_{*}**.

The next theorem is from [2] and states that being non U-shaped is not restrictive for **TxtEx**-learning. Its proof can also be obtained by letting $b = 1$ in Theorem 12 below. Its fundamental equality will be extended to all classes **NUShTxtFex_b** in Theorem 7.

Theorem 5. [2] **NUShTxtFex₁** = **TxtEx**.

Hence, for \mathbf{TxtEx} , U-shaped behaviour is not necessary for full learning power. By contrast, an easy adaptation of the proof of Theorem 4 in [16] shows that, for \mathbf{TxtBc} , U-shaped behaviour is necessary for full learning power.

Theorem 6. [2, 16] $\mathbf{NUShTxtBc} \subseteq \mathbf{TxtBc}$.

3 Non U-shaped vacillatory learning

We show that the \mathbf{TxtFex}_b -hierarchy collapses if U-shaped behaviour is forbidden.

Theorem 7. $\mathbf{NUShTxtFex}_* \subseteq \mathbf{TxtFex}_1$.

Theorems 5 and 7 give $\mathbf{NUShTxtFex}_1 = \mathbf{TxtFex}_1$ and $\mathbf{NUShTxtFex}_* \subseteq \mathbf{NUShTxtFex}_1$. Furthermore, the definition of $\mathbf{NUShTxtFex}_b$ immediately gives $\mathbf{NUShTxtFex}_1 \subseteq \mathbf{NUShTxtFex}_b \subseteq \mathbf{NUShTxtFex}_*$. Thus all these criteria coincide.

Corollary 8. $(\forall b \in \{1, 2, \dots, *\}) [\mathbf{NUShTxtFex}_b = \mathbf{TxtFex}_1]$.

The result $\mathbf{NUShTxtFex}_1 = \mathbf{TxtFex}_1$ stands in contrast to the fact that $\mathbf{TxtFex}_1 \subseteq \mathbf{TxtFex}_2 \subseteq \dots \subseteq \mathbf{TxtFex}_*$. Thus we have that the following inclusions are proper.

Corollary 9. $(\forall b \in \{2, 3, \dots, *\}) [\mathbf{NUShTxtFex}_b \subset \mathbf{TxtFex}_b]$.

Corollaries 8 and 9 show that U-shaped learning behaviour is *necessary* for the full learning power of \mathbf{TxtFex}_b -identification for $b > 1$ in a strong sense: if U-shaped learning behaviour is forbidden, the hierarchy collapses to \mathbf{TxtFex}_1 . Hence, the \mathbf{TxtFex}_* -learnability of any class in $(\mathbf{TxtFex}_* - \mathbf{TxtFex}_1)$ *requires* U-shaped learning behaviour.

Corollary 10. *Let $b \in \{2, 3, \dots, *\}$ and \mathcal{H}_b be the class from Theorem 3. Then any machine witnessing $\mathcal{H}_b \in \mathbf{TxtFex}_b$ necessarily employs U-shaped learning behaviour on \mathcal{H}_b .*

A non U-shaped learner does not make a mind change from a correct to an incorrect hypothesis since it cannot learn the set otherwise. This property is enforced on all machines for the case of team learning.

Definition 11. A team learning a class \mathcal{L} is *non U-shaped* iff no machine in the team on any text for any language $L \in \mathcal{L}$ ever makes a mind change from an index for L to an index for a different language. In particular, the class \mathcal{L} is in $[a, b]\mathbf{NUShTxtEx}$ iff there are b machines such that on any text for any language in \mathcal{L} at least a machines converge to an index for that language and no machine makes a mind change from a correct to an incorrect hypothesis. For any learning criterion I , $[a, b]I$ is the corresponding team variant of this criterion.

The next result shows that Theorem 2 can be extended such that every class in \mathbf{TxtFex}_b is learnable from a non U-shaped team. So the restriction $\mathbf{NUShTxtFex}_b = \mathbf{TxtFex}_1$ is caused by the fact that the hypothesis of the learner have to be brought into an ordering and cannot be done in parallel as in the case of the team below. Actually Theorem 12 enables us to achieve more properties of the team than that it is just non U-shaped.

Theorem 12. *Let $b \in \mathbb{N}^+$ and $\mathcal{L} \in \mathbf{TxtFex}_b$. Then there is a team of b learners $\mathbf{M}_1, \dots, \mathbf{M}_b$ such that for all $L \in \mathcal{L}$ and all texts T for L there is an $n \in \mathbb{N}$ such that,*

- $T[n]$ is a stabilizing sequence of all members of the team on L , in particular $\mathbf{M}_a(T[m]) = \mathbf{M}_a(T[n])$ for all $m \geq n$;
- there is an $a \in \{1, \dots, b\}$ such that $\mathbf{M}_a(T[n])$ is an index for L ;
- if $a \in \{1, \dots, b\}$ and $\mathbf{M}_a(T[m])$ is an index for L then $m \geq n$.

In particular, $\mathbf{M}_1, \dots, \mathbf{M}_b$ $[1, b]$ **NUShTxE**-learns \mathcal{L} .

The next result gives a further inclusion between vacillatory learning and team-learning.

Theorem 13. (a) $\mathbf{TxFex}_b \subseteq [2, b+1]$ **NUShTxE** for all $b \in \{1, 2, 3, \dots\}$.
(b) $\mathbf{TxFex}_b \subseteq [1, b]$ **NUShTxE** for all $b \in \{2, 3, \dots\}$.

As \mathbf{TxFex}_2 -learning is more general than **TxE**-learning, one can conclude that also $[2, 3]$ **TxE**-learning is strictly more general than **TxE**-learning.

Corollary 14. $[2, 3]$ **NUShTxE** $\not\subseteq$ **NUShTxE**.

4 Vacillatory Learning with 2 Indices

As every $\mathbf{NUShTxFex}_b$ -learner can be turned into a **NUShTxE**-learner identifying the same class, the restriction to *non* U-shaped learning without loss of learning power is only possible in the least class \mathbf{TxFex}_1 of the \mathbf{TxFex}_b hierarchy. But, the next, quite surprising result shows that in the case of \mathbf{TxFex}_2 one can avoid U-shaped learning behaviour if one gives up the constraint that the learner has to vacillate between *finitely* many indices. That is, $\mathbf{TxFex}_2 \subseteq \mathbf{NUShTxBc}$. In Theorem 16 it is shown that there is a uniform learner \mathbf{U} which is given a set E of up to 2 indices and **NUShTxBc**-identifies every $\{W_e : e \in E\}$ such that every hypothesis is a subset of an W_e with $e \in E$. Then this result is combined with Theorem 12 to show the inclusion $\mathbf{TxFex}_2 \subseteq \mathbf{NUShTxBc}$. But before turning to Theorem 16, the following auxiliary proposition gives a method to enforce that the sets $W_{i'}, W_{j'}$ mentioned there are represented by corresponding approximations W_i, W_j which differ from one another at all *relevant* stages of their enumerations.

Proposition 15. Given a set $F = \{i', j'\}$ one can compute a set $G(F) = \{i, j\}$ such that

- For all s , either $W_{i,s} \cup W_{j,s} = \emptyset$ or $W_{i,s} \neq W_{j,s}$;
- $W_i \subseteq W_{i'}$ and if $W_{i'}$ is infinite then $W_i = W_{i'}$;
- $W_j \subseteq W_{j'}$ and if $W_{j'}$ is infinite then $W_j = W_{j'}$;
- $\{W_{i'}, W_{j'}\} \subseteq \{W_i, W_j\}$.

This also holds with $j' = i'$ for the case that $F = \{i'\}$.

Theorem 16. There is a learner \mathbf{U} such that for all r.e. sets L, H and every set F of indices for L, H with $|F| \leq 2$, \mathbf{U} **NUShTxBc**-identifies $\{L, H\}$ using the additional information F . Furthermore, for every $\sigma \in \text{seg}(\mathbb{N})$,

- (1) $W_{\mathbf{U}(F, \sigma)} \subseteq L$ or $W_{\mathbf{U}(F, \sigma)} \subseteq H$;
- (2) if $L = H$ and L is infinite then $W_{\mathbf{U}(F, \sigma)} \in \{\emptyset, L\}$.

Note that $L = H$ is explicitly permitted.

Uniform non U-shaped Behaviourally Correct Learner U

Parameter: F . Input: σ . Output: k , specified implicitly.

Algorithm to enumerate $W_k = \bigcup_u W_{k,u}$.

(Start) Let $u = |\sigma|$, $C = \text{content}(\sigma)$ and $s = 0$.

Let $W_{k,t} = \emptyset$ for all $t < |\sigma|$.

If $C = \emptyset$ or $W_{e,u} = \emptyset$ for all $e \in G(F)$ Then go to (Empty).

Let $\tau = \sigma[|\sigma| - 1]$.

Select i, j, x such that

(a) $\{i, j\} = G(F)$;

(b) $x = \min(W_{i,u} \Delta W_{j,u})$;

(c) $x \in W_{i,u} \Leftrightarrow x \in C$.

Go to (Branch).

(Branch) If $C \cup W_{i,s} \subseteq W_{\mathbf{U}(F,\tau),u}$ and $y \in (W_{i,u} - W_{i,|\sigma|}) \cup (W_{j,u} - W_{j,|\sigma|})$ for some $y \leq x$ Then go to (Copy) Else go to (Enum).

(Enum) Let t be the maximal element of $\{s, \dots, u\}$ such that one of the following conditions holds:

(Min) $t = s$;

(Inf) $C \subseteq W_{i,t} \subset W_{i,u} \wedge \forall y \leq x (y \notin (W_{i,u} - W_{i,|\sigma|}) \cup (W_{j,u} - W_{j,|\sigma|}))$;

(Diff) $C \subseteq W_{i,t} \subset W_{i,u}$ and $W_{j,u} = W_{i,s}$;

(Equal) $C \subseteq W_{i,t} \cup W_{j,t} \subset W_{i,u} \cap W_{j,u}$;

(Sub) $C \subseteq W_{i,t} \subseteq W_{\mathbf{U}(F,\tau),u}$;

(Exact) $C = W_{i,t}$ and $t = |\sigma|$.

Let $W_{k,u} = W_{i,t}$, update $s = t$, $u = u + 1$ and go to (Branch).

(Copy) Let $W_{k,u} = W_{\mathbf{U}(F,\tau),u}$, update $u = u + 1$ and go to (Copy).

(Empty) Let $W_{k,u} = \emptyset$, update $u = u + 1$ and go to (Empty).

Fig. 1. Algorithm to enumerate $W_{\mathbf{U}(F,\sigma)}$ from Theorem 16.

Given F , let $G(F)$ as in Proposition 15 and let Δ denote the symmetric difference of sets, that is, $A\Delta B = (A \cup B) - (A \cap B)$. Figure 1 gives the algorithm witnessing this inclusion. We omit the proof that it works.

Theorem 17. *Every TxtFex_2 -learnable class is NUShTxtBc -learnable.*

Proof. Given a TxtFex_2 -learnable class \mathcal{L} , there is by Theorem 2 a pair of two learners $\mathbf{N}_1, \mathbf{N}_2$ which converge on every language from \mathcal{L} and $[1, 2]\text{TxtEx}$ -identify \mathcal{L} . Obtain from these two learners the team $\mathbf{M}_1, \mathbf{M}_2$ as done in the proof of Theorem 12. Note that one can, for each input σ , check in the limit whether one of the learners $\mathbf{M}_1, \mathbf{M}_2$ outputs an index which enumerates some elements using part (f) of the algorithm in the proof of Theorem 12. Now one builds the following new learner \mathbf{U}' :

$W_{\mathbf{U}'(\tau)} = W_{\mathbf{U}(\{e_1, e_2\}, \tau)}$ where, for $a = 1, 2$,

$$W_{e_a} = \begin{cases} W_{\mathbf{M}_a(\tau)} & \text{if (f) is not used for } W_{\mathbf{M}_1(\tau)} \text{ or } W_{\mathbf{M}_2(\tau)}; \\ W_{\mathbf{M}_1(\tau)} \cup W_{\mathbf{M}_2(\tau)} & \text{if (f) is used for } W_{\mathbf{M}_1(\tau)} \text{ or } W_{\mathbf{M}_2(\tau)}. \end{cases}$$

Let $L \in \mathcal{L}$ and T be a text for L . Let n be the first number where one of the sets $W_{\mathbf{M}_1(T[n])}, W_{\mathbf{M}_2(T[n])}$ is L . Then, for any $m \geq n$ and any $a \in \{1, 2\}$, $W_{\mathbf{M}_a(T[m])} = W_{\mathbf{M}_a(T[n])}$ and $W_{\mathbf{M}_a(T[m])}$ is not of the form $E \cup \tilde{E}$ where \tilde{E} is finite. Then \mathbf{U} is fed with the same parameter set $\{e_1, e_2\}$ for all $m \geq n$ and one of the e_1, e_2 enumerates L . Thus \mathbf{U}' TxtBc -learns L on T .

It remains to show that \mathbf{U}' is non U-shaped on T . This is clearly true if L is the empty set. So assume $L \neq \emptyset$. Consider any m with $W_{\mathbf{U}'(T[m])} = L$. If case (f) of the algorithm for $\mathbf{M}_1(T[m])$ and $\mathbf{M}_2(T[m])$ applies, then W_{e_1} and W_{e_2} are the same infinite set $E \cup \tilde{E}$ for some finite set \tilde{E} . It follows by the additional property (2) of \mathbf{U} in Theorem 16 that $\mathbf{U}'(T[m])$ either outputs

an index for the empty set or for $E \cup \tilde{E}$; both sets are different from L , thus case (f) does not apply. Hence, $T[m]$ is a stabilizing sequence for both $\mathbf{M}_1, \mathbf{M}_2$ on those sets $W_{\mathbf{M}_1(T[m])}, W_{\mathbf{M}_2(T[m])}$ which are not empty. Since one of these is a superset of L by the additional property (1) of \mathbf{U} in Theorem 16, it follows that $\mathbf{M}_1, \mathbf{M}_2$ do not change mind on T beyond $T[m]$. For $a = 1, 2$ the parameter e_a is defined as above for $\tau = T[m]$ and it holds that $W_{e_a} = W_{\mathbf{M}_a(T[m])}$. Thus $\mathbf{U}'(T[o])$ coincides with $\mathbf{U}(\{e_1, e_2\}, T[o])$ for all $o \geq m$ and \mathbf{U} with the parameter set $\{e_1, e_2\}$ is non U-shaped on the text T for L . The same holds for \mathbf{U}' . Thus \mathbf{U}' **NUShTxtBc**-learns \mathcal{L} . ■

5 Vacillatory Learning with 3 Indices

From Theorem 7 it is already known that U-shaped learning behaviour is necessary for **TxtFex_b** ($b > 1$) identification of any class in **TxtFex_b** – **TxtFex₁** for all $b > 1$. Theorem 18 strengthens this result by showing that, for some classes of languages in **TxtFex_b** for $b > 2$, the necessity of U-shaped behaviour cannot be circumvented by allowing infinitely many correct grammars in the limit, that is, by shifting to the more liberal criterion of **TxtBc**-identification. This is one of the rare cases in inductive inference where the containment in a class defined without numerical parameters holds for level 2 but not for level 3 and above of a hierarchy. The proof is a diagonalization proof reminiscent of the proof of Theorem 4 in [16].

Theorem 18. **TxtFex₃ $\not\subseteq$ NUShTxtBc.**

Proof. Let $L_{i,j} = \{\langle i, j, k \rangle : k \in \mathbb{N}\}$, $I_{i,j} = W_i \cap L_{i,j}$ and $J_{i,j} = W_j \cap L_{i,j}$ for $i, j \in \mathbb{N}$. The class

$$\mathcal{L} = \{L_{i,j} : i, j \in \mathbb{N}\} \cup \{I_{i,j}, J_{i,j} : i, j \in \mathbb{N} \wedge I_{i,j} \subset J_{i,j} \wedge |I_{i,j}| < \infty\}$$

witnesses the separation.

To see that \mathcal{L} is in **TxtFex₃**, consider the following machines $\mathbf{N}_I, \mathbf{N}_J, \mathbf{N}_L$ which initially output indices of the empty set. Each of them waits for the first tuple of the form $\langle i, j, k \rangle$ for some k to come up in the input. From then on, \mathbf{N}_I outputs an index for $I_{i,j}$ forever, \mathbf{N}_J an index for $J_{i,j}$ forever and \mathbf{N}_L an index for $L_{i,j}$ forever. So, for every $i, j \in \mathbb{N}$, \mathbf{N}_I learns the set $I_{i,j}$, \mathbf{N}_J the set $J_{i,j}$ and \mathbf{N}_L the set $L_{i,j}$. The class \mathcal{L} is learnable by a team of three machines which converge on every text for every language in \mathcal{L} to some index. It follows from Theorem 2 that \mathcal{L} is in **TxtFex₃**.

So it remains to show that \mathcal{L} is not in **NUShTxtBc**, that is, to show that any given **TxtBc**-learner for \mathcal{L} is U-shaped on some text for some language in \mathcal{L} . Given the learner \mathbf{M} , one defines the following function F by an approximation from below:

$$F_s(i, j) = \begin{cases} F_{s-1}(i, j) & \text{if } s > 0 \text{ and} \\ & W_{\mathbf{M}(\langle i, j, 0 \rangle \langle i, j, 1 \rangle \dots \langle i, j, F_{s-1}(i, j) \rangle)}, s \subseteq L_{i,j}; \\ k & \text{otherwise where } k \text{ is the first number} \\ & \text{found with } k > F_{s-1}(i + j) + s \text{ and} \\ & \{\langle i, j, 0 \rangle, \langle i, j, 1 \rangle, \dots, \langle i, j, k \rangle\} \subset \\ & W_{\mathbf{M}(\langle i, j, 0 \rangle \langle i, j, 1 \rangle \dots \langle i, j, k \rangle)}. \end{cases}$$

Since $\langle i, j, 0 \rangle, \langle i, j, 1 \rangle, \dots$ is a text for $L_{i,j}$ and \mathbf{M} **TxtBc**-learns $L_{i,j}$, almost all hypotheses $\mathbf{M}(\langle i, j, 0 \rangle \langle i, j, 1 \rangle \dots \langle i, j, k \rangle)$ are indices for $L_{i,j}$. Thus the k is always found in the second part of the definition of F_s and F_s is well-defined. Furthermore, if $F_{s-1}(i, j)$ is sufficiently large,

the condition $W_{\mathbf{M}(\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F_{s-1}(i,j) \rangle),s} \subseteq L_{i,j}$ holds for all s and thus $F_s(i,j) = F_{s-1}(i,j)$. So the limit $F(i,j)$ of all $F_s(i,j)$ exists and is approximated from below. By considering the first s where $F(i,j) = F_s(i,j)$ and the fact that it is then no longer updated, one has

$$\{\langle i,j,0 \rangle, \langle i,j,1 \rangle, \dots, \langle i,j,F(i,j) \rangle\} \subset W_{\mathbf{M}(\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F(i,j) \rangle)} \subseteq L_{i,j}.$$

Now there are r.e. sets W_a, W_b such that

$$\begin{aligned} W_a &= \{\langle i,j,l \rangle : i,j \in \mathbb{N} \wedge l \in \{0,1,\dots,F(i,j)\}\}, \\ W_b &= \{\langle i,j,l \rangle : i,j \in \mathbb{N} \wedge \exists t > l (\langle i,j,l \rangle \in W_{\mathbf{M}(\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F_t(i,j) \rangle),t})\}. \end{aligned}$$

Now fix the parameters i,j such that $i = a$ and $j = b$; the cases where $i \neq a$ or $j \neq b$ are not important in the below considerations.

Assume that $\langle i,j,l \rangle \in W_j$ using a parameter t with $F_t(i,j) \neq F(i,j)$. Let s be the first stage with $F_s(i,j) = F(i,j)$; note that $s > t$. Then by the definition of F_s , $F(i,j) = F_s(i,j) > s > t$ and $\{\langle i,j,0 \rangle, \langle i,j,1 \rangle, \dots, \langle i,j,F_s(i,j) \rangle\} \subset W_{\mathbf{M}(\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F_s(i,j) \rangle)}$. So $\langle i,j,l \rangle \in W_{\mathbf{M}(\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F(i,j) \rangle)}$ as well.

Thus $\{\langle i,j,0 \rangle, \langle i,j,1 \rangle, \dots, \langle i,j,F(i,j) \rangle\} = W_i \cap L_{i,j} = I_{i,j} \subset J_{i,j} = W_j \cap L_{i,j} = W_{\mathbf{M}(\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F(i,j) \rangle)}$ and $I_{i,j}$ is finite. Hence $I_{i,j}, J_{i,j} \in \mathcal{L}$.

There is a text T for $J_{i,j}$ which starts with $\langle i,j,0 \rangle \langle i,j,1 \rangle \dots \langle i,j,F(i,j) \rangle$ and after which \mathbf{M} outputs an index for $J_{i,j}$. Then a lot of pause symbols follow until \mathbf{M} outputs an index for $I_{i,j}$. After that the remaining elements of $J_{i,j}$ follow and \mathbf{M} has to return to the index $J_{i,j}$. So \mathbf{M} is not a **NUShTxtBc**-learner for \mathcal{L} . Since \mathbf{M} was chosen arbitrarily, \mathcal{L} is not **NUShTxtBc**-learnable. ■

Since $\mathbf{TxtFex}_3 \subset \mathbf{TxtFex}_4 \subset \dots \subset \mathbf{TxtFex}_*$, one immediately gets the following corollary.

Corollary 19. $(\forall b \in \{3,4,\dots,*\}) [\mathbf{TxtFex}_b \not\subseteq \mathbf{NUShTxtBc}]$.

A further corollary is that the counterpart of Theorem 16 does not hold for sets of three indices. Indeed, if such an algorithm would exist, then one could **NUShTxtBc**-learn \mathcal{L} from Theorem 18 by conjecturing \emptyset until the first triple $\langle i,j,k \rangle$ comes up and then simulating the uniform learner with a set of three indices for the sets $I_{i,j}, J_{i,j}, L_{i,j}$ from then on without changing this parameter set anymore. But Theorem 18 clearly showed that such a learner does not exist.

Corollary 20. *No machine uniformly **NUShTxtBc**-learns $\{W_e : e \in F\}$ with F as additional information where F is a set of 3 indices.*

6 Teams Revisited

Classes in \mathbf{TxtFex}_2 are in \mathbf{TxtBc} and in $[1,2]\mathbf{NUShTxtEx}$. The next proposition shows that one cannot weaken the condition of being in \mathbf{TxtFex}_2 to the combination of the two consequences in Theorem 17. Furthermore the condition that the team members converge on every text for a language in \mathcal{L} is essential in Theorem 2.

Proposition 21. *The class \mathcal{L} from Theorem 18 is $[1,2]\mathbf{NUShTxtEx}$ -learnable and \mathbf{TxtFex}_3 -learnable but it is not **NUShTxtBc**-learnable.*

Remark 22. $\mathbf{TxtFex}_* \not\subseteq [1, b]\mathbf{TxtEx}$ for all $b \in \mathbb{N}^+$. This is witnessed by the class $\{W_e : e \in \mathbb{N} \wedge \{0, 1, \dots, e\} \subseteq W_e \subset \mathbb{N}\}$. Note that by Proposition 21 it can be that a class in $\mathbf{TxtFex}_{b+1} - \mathbf{TxtFex}_b$ is already $[1, b]\mathbf{TxtEx}$ -learnable.

A further interesting question is whether one can at least obtain non U-shaped team learning for arbitrary team learnable classes. This is true for $[1, 1]\mathbf{TxtEx}$ by Theorem 12 but it fails for $[1, 2]\mathbf{TxtEx}$ -learning.

Theorem 23. For all $b \in \{2, 3, \dots\}$, $[1, b]\mathbf{NUShTxtEx} \subset [1, b]\mathbf{TxtEx}$.

Proposition 24. $[a, b]\mathbf{TxtEx} \subseteq [a, a + b]\mathbf{NUShTxtEx}$.

7 Summary and Final Discussion

The following results were obtained.

- $\mathbf{TxtFex}_b \subset [1, b]\mathbf{NUShTxtEx}$ for all $b \in \{2, 3, \dots\}$.
- $\mathbf{TxtFex}_1 = \mathbf{TxtEx} = \mathbf{NUShTxtEx} = [1, 1]\mathbf{NUShTxtEx}$, see also [2].
- $[1, b]\mathbf{NUShTxtEx} \subset [1, b]\mathbf{TxtEx}$, for all $b \in \{2, 3, \dots\}$.
- $\mathbf{NUShTxtFex}_b = \mathbf{NUShTxtEx}$ for all $b \in \{1, 2, \dots, *\}$.
- $\mathbf{TxtFex}_2 \subseteq \mathbf{NUShTxtBc}$.
- $\mathbf{TxtFex}_3 \not\subseteq \mathbf{NUShTxtBc}$.

Some of the results obtained herein and mentioned just above are *suggestive* for what may be true for the case of human cognition. For example, *perhaps* the class of tasks humans must learn to be competitive in the genetic marketplace like, say, our class $\mathcal{L} \in (\mathbf{TxtFex}_3 - \mathbf{NUShTxtBc})$, *necessitates* U-shaped learning behaviour. In the human context, U-shaped learning *may* be necessary.

We note that our proof that $\mathbf{TxtFex}_3 \not\subseteq \mathbf{NUShTxtBc}$ intriguingly features learning finite tables versus general rules, but does *not*, as might be expected from some models of the human case of U-shaped learning, feature, among other things, learning an incorrect general rule followed by learning a general rule augmented by a correcting finite table. This difference may be significant or, more likely, nothing more than an artifact of our particular proof.

Not explored herein, but very interesting to investigate in the future, is the possible necessity of U-shaped learning in complexity-bounded learning contexts such as those explored in [10, 11, 14, 28, 33].

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