Worms, Gaps and Hydras

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Abstract

We define a direct translation from finite rooted trees to finite natural functions which shows that the Worm Principle introduced by Lev Beklemishev is equivalent to a very slight variant of the well-known Kirby-Paris' Hydra Game. We further show that the elements in a reduction sequence of the Worm Principle determine a bad sequence in the well-quasi-ordering of finite sequences of natural numbers with respect to Friedman's gap-embeddability.

1 Introduction

With his recent work on Graded Provability Algebras (see [1]), Lev Beklemishev has proposed a new promising approach to the problem of *natural well-orderings*, a conceptual problem underlying contemporary ordinal analysis. His work has shown how an ordinal notation system for ε_0 can be *canonically* extracted from the structure of the Graded Provability Algebra of Elementary Arithmetic ($\mathbf{EA}$), i.e. from the Lindenbaum Algebra of $\mathbf{EA}$ enriched with n-consistency operators $\langle n \rangle$ for each n. Whether this method can be applied successfully to systems stronger than $\mathbf{PA}$, and especially to impredicative systems, is still unknown. As usual, a new approach to ordinal analysis results in some new independent principle, and Beklemishev's approach is no exception. The so called Worm Principle, a rewriting game on finite words of natural numbers, whose termination is provably equivalent to the 1-consistency of Peano Arithmetic ($\mathbf{PA}$), is directly obtained from the new provability-algebraic notation system. It came as a bit of a surprise that, as independently observed by G. Lee, A.Weiermann and the present author (see [2]), the system of fundamental sequences obtained in this way can be shown to be a slight variant of the standard one used in proof theory, providing a provability-algebraic justification of the latter.

The aim of this note is to present two simple relationships between Beklemishev's Worm Principle [2], Kirby-Paris' Hydra Game [9], and Schütte-Simpson's one- dimensional version of the Friedman-Kruskal Theorem [11]. Both comparisons between those $\mathbf{PA}$-unprovable theorems are established in a very direct and simple way and with no use of ordinal notations (in the spirit of [5]), a fact that makes the present paper suitable for expository purposes. The author also hopes that these results may shed some light on the principles involved and, thanks to their very elementary character,

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be of interest to the non specialist, who is often refrained by the heavy use of ordinal notations.

The relationship between the Worm Principle and the Schütte-Simpson Theorem is particularly interesting in the author's opinion since it is well-known that the notion of gap-embeddability was introduced by Friedman in a quite *ad hoc* way to obtain independence results from strong systems. The connection with the Worm Principle and therefore with provability algebras is quite surprising and provides the first independent justification of this concept. In this respect, we also present a characterization of gap-embeddability in terms of implication in a system of provability logic. This result has been obtained by Lev Beklemishev after seeing our Theorem 3, and is presented here with his kind permission. This result shows in an even stronger way how gap-embeddability naturally emerges from provability logic and graded provability algebras.

At the end of each section some comments are made concerning the the fine tuning à la Weiermann of the principles involved. Similar results have been independently obtained by Gyesik Lee and Andreas Weiermann in unpublished work, with heavy use of tree-ordinals and ordinal notation systems.

2 Worms and Hydras

In this section we define a direct translation from hydras to worms which shows that the Hydra Game and the Worm Principle correspond *almost* step-by-step. An exact step-by-step correspondence is obtained by very slighly modifying the Hydra Game or, equivalently, the standard assignment of fundamental sequences below ε_0. The latter fact has been independently observed by Weiermann and Lee (see [2]). The translation we give here to establish the correspondence is of a "combinatorial" flavour and differs both from the natural mapping between worms and ordinals smaller than ε_0 defined by Beklemishev in [1] and from the mapping used by Weiermann and Lee.

A *worm* is a function $f : [0,n] \longrightarrow \mathbf{N}$. Worms can be conveniently treated as strings or words. The Worm Principle is defined by the following rules. Define function $\mathsf{next}(w,m)$ as follows. If $f(n) = 0$ then $\mathsf{next}(w,m) := (f(0),\dots,f(n-1))$. If $f(n) > 0$ let $k := \max_{i<n} \; f(i) < f(n)$; let $r := (f(0),\dots,f(k))$, and $s := (f(k+1),\dots,f(n-1),f(n)-1)$, then set $\mathsf{next}(w,m) := r * s * \cdots * s$ with $m+1$ copies of s, $*$ denoting concatenation.

For a worm w, let $w_0 = w$ and let $w_{n+1} = \mathsf{next}(w_n, n+1)$. Let EWD (Every Worm Dies) be the following sentence in the language of $\mathbf{PA}$.

$$(\forall w)(\exists n)[w_n = \varnothing].$$

Beklemishev proved in [1] that EWD is equivalent to the 1-Consistency of $\mathbf{PA}$ over primitive recursive arithmetic, and therefore that EWD is independent from $\mathbf{PA}$.

We refer to ordered finite rooted trees simply as trees. Call *level* of a node ν in a tree T (denoted by $lvl_T(\nu)$) the following measure: the root has level -1, the successors of the root have level 0, and so on. That is, the level of a node in a tree is smaller by one unit than its usual height. Let h_1, h_2 be two heads (leaves) of a tree T. Then by $h_1 \sqcap h_2$ we indicate the first (nearest) common ancestor of h_1 and h_2. Let ϵ denote the empty sequence. *Hydras* are finite rooted trees (referred to simply as trees from now on).

Definition 1. We define a translation $* : Hydras \to Worms$ as follows. Let $h_1, \dots h_n$ be the heads (leaves) of the tree T in the visit order of T. Then we set

$$T^* := \langle lvl(h_1) a_1 \dots a_{n-1} lvl(h_n) \rangle$$

where

$$a_i = \begin{cases} lvl(h_i \sqcap h_{i+1}) & \text{if } lvl(h_i \sqcap h_{i+1}) < \min\{lvl(h_i) - 1, lvl(h_{i+1}) - 1\} \\ \epsilon & \text{otherwise.} \end{cases}$$

An intuitive way to look at the translation is the following. Take a tree T. Write down the levels of the leaves of T in the visiting order. From left to right, for each couple of consecutive leaves, consider their first common ancestor. If the latter occurs at a level smaller than the level of the lower between the two leaves minus 1, then insert the value of this level between the values of the two leaves, otherwise insert nothing.

Recall for convenience the rules of Kirby-Paris' Hydra Game from [9].

At stage n (n a positive integer), Hercules chops one head h of the hydra $\mathcal{H}$. If the predecessor of h is the root, nothing happens. Otherwise let h^1 and h^2 be respectively the father and the grandfather of h. The hydra sprouts n copies of the principal subtree determined by h^1 without the head h from the node h^2 (the the roots of the new copies are immediate successors of the node h^2). Hercules wins the battle if he reduces in a finite number of attacks the hydra to its root.

With Definition 1 in hand we can prove the following. Let $\mathsf{red}(T, n)$ denote the tree obtained from tree T by one step of the Hydra Game starting at stage n using the strategy *always cut the rightmost head*. Let $\sqsubset$ denote the prefix relation. If w is a worm $\langle w_1 \dots w_n \rangle$ let $S(w)$ (the *successor* of σ) denote the worm $\langle w_1 + 1 \dots w_n + 1 \rangle$. Let $\hookrightarrow_W$ denote the one-step rewrite relation of the Worm Principle and $\hookrightarrow_H$ denote the one-step rewrite relation of the Hydra Game with the strategy *always cut the rightmost head*.

Theorem 1. $(\forall n)[\mathsf{red}(T, n)^* \sqsubset \mathsf{next}(T^*, n)]$.

Proof. By induction on the structure of trees we have three cases (corresponding to the cases $\beta + 1$, $\gamma + \omega^{\beta+1}$, $\gamma + \omega^\lambda$ with λ limit, on ordinals less than ε_0).

Case 1. T is of the form: a subtree T_1 plus one head joined to the root. By definition of *, we have $T^* = T_1^* 0$. By the rules of the Hydra Game, we have the reduction step in Figure 1: the rightmost head disappears.

Therefore $\mathsf{red}(T, n) = T_1$, and $\mathsf{red}(T, n)^* = T_1^*$.

On the other hand, by the rules of the Worm Principle, we have:

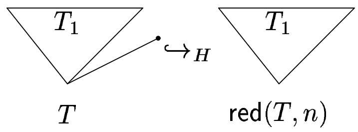


Figure 1: Case 1

$$T^* = T_1^*0 \hookrightarrow_W T_1^* = \mathsf{next}(T^*, n)$$

that is, $\mathsf{next}(T^*, n) = \mathsf{next}(T_1^*0, n) = T_1^*$. Therefore we have $\mathsf{red}(T, n)^* = \mathsf{next}(T^*, n)$.

Case 2. T has two maximal subtrees T_1 and T_2 joined at the root. T_2 has the same form of T in Case 1 but lifted one level up: one arc then one subtree T_3 and one head sprouting on the right of the root of T_3. By definition of *, $T^* = T_1^*0T_2^* = T_1^*0S(T_3^*0) = T_1^*0S(T_3^*)1$. By the rules of the Hydra Game, T undergoes the reduction step shown in Figure 2: $n+1$ copies of T_3 without its rightmost head are created.

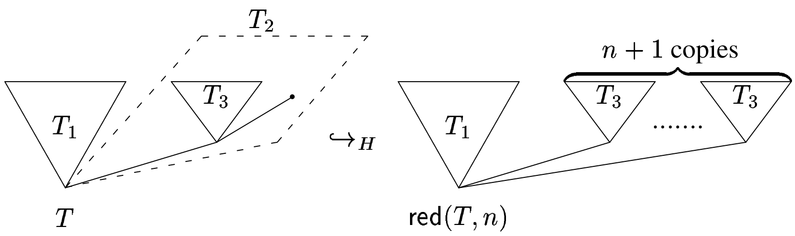


Figure 2: Case 2

Therefore $\mathsf{red}(T, n)$ is formed by T_1 joined at the root with $n+1$ copies of T_2 *without* the distinguished rightmost head sprouting from the root of T_3. Therefore $\mathsf{red}(T, n)^* = T_1^* \underbrace{0S(T_3^*) \dots 0S(T_3^*)}_{n+1 \ copies \ of \ 0S(T_3^*)}$.

On the other hand, since in $S(T_3^*)$ there can be no element smaller than 1, we have, by the rules of the Worm Principle:

$$T_1^*0S(T_3^*)1 \hookrightarrow_W T_1^*0 \underbrace{S(T_3^*)0 \dots S(T_3^*)0}_{n+1 \ copies \ of \ S(T_3^*)0} = T_1^* \underbrace{0S(T_3^*) \dots 0S(T_3^*)}_{n+1 \ copies \ of \ 0S(T_3^*)} 0 = \mathsf{next}(T^*, n)$$

Therefore

$$\mathsf{red}(T,n)^* = T_1^* \underbrace{0S(T_3^*)\dots 0S(T_3^*)}_{n+1 \text{ copies of } 0S(T_3^*)} \sqsubset T_1^* \underbrace{0S(T_3^*)\dots 0S(T_3^*)}_{n+1 \text{ copies of } 0S(T_3^*)} 0 = \mathsf{next}(T^*,n).$$

Case 3. T is composed of a two subtrees T_1, T_2 joined at the root. The only thing we know about T_2 is that it starts with one arc and then has a maximal subtree T_3 (with no level 0 heads) sprouting from that arc at level 0. By definition of *, $T^* = T_1^* 0S(T_3^*)$. T undergoes the Hydra Game reduction shown in Fig. 3:

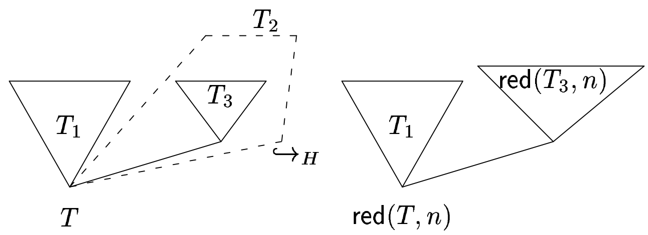


Figure 3: Case 3

by the rules of the Hydra Game, the only part affected in T is T_3 (since it has no head sprouting from its root), and therefore $\mathsf{red}(T,n)$ is just T with T_3 changed to $\mathsf{red}(T_3,n)$.

By the inductive hypothesis, $\mathsf{red}(T_3,n)^* \sqsubset \mathsf{next}(T_3^*,n)$. By the Worm Principle we have:

$$T^* = T_1^* 0S(T_3^*) \hookrightarrow_W T_1^* 0S(\mathsf{next}(T_3^*,n)) = \mathsf{next}(T^*,n).$$

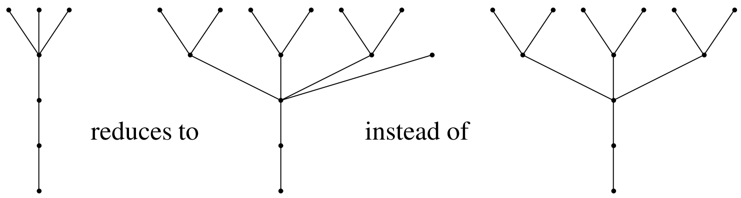
Therefore $\mathsf{red}(T,n)^* = T_1^* 0S(\mathsf{red}(T_3,n)^*) \sqsubset T_1^* 0S(\mathsf{next}(T_3^*,n)) = \mathsf{next}(T^*,n)$. □

Corollary 2. *T^* rewrites to $\mathsf{red}(T,n)^*$ for each n by the rules of the Worm Principle.*

Proof. Beklemishev ([1]) observes that if $v \sqsubset w$ then w rewrites to v in a finite number of steps of the Worm Principle. Obviously T^* rewrites to $\mathsf{next}(T^*,n)$ for each n. □

This shows that termination of the Worm Principle implies termination of the Hydra Game. The independence of EWD from **PA** follows.

Remarks on the fine tuning of the Worm Principle It is easy to see from Theorem 1 that the Worm Principle corresponds step by step to a slight modification of the Hydra Game, in which one more head is adjoined (in rightmost position) to the grandfather of the head which is cut. For example:



Besides, if one considers trees in Cantor Normal Form, the strategy *always cut the rightmost head* is equivalent to Kirby-Paris' strategy.

If one modifies the standard assignement of fundamental sequences to ordinals less than ε_0 as follows:

$$(\beta{+}1)[n] = \beta,\ (\gamma{+}\omega^{\beta+1})[n] = \gamma{+}\omega^{\beta}\times(n{+}1){+}1,\ (\gamma{+}\omega^{\lambda})[n] = \gamma{+}\omega^{\lambda[n]},\ \text{for } \lambda \text{ limit}$$

then the corresponding Hardy functions measure the lengths of battles of the modified Hydra Game (with Kirby-Paris' strategy), and therefore of the Worm Principle.

We now make some remarks on the fine tuning à la Weiermann for the Worm Principle.

Weiermann has completely characterized the growth functions for which the termination of the Hydra Game is unprovable in Peano Arithmetic (unpublished, announced in [12]). Let HG_f indicate the Hydra Game in which at step n, the number of replicas produced is $f(n)$, where f is some natural function. Consider, for $\alpha \leq \varepsilon_0$, the function $f_\alpha(i) := |i|_{H_\alpha^{-1}(i)}$, where $|\cdot|$ denotes the binary length function and $H_\alpha^{-1} := \mu\{k | H_\alpha(k) \geq i\}$. Then the termination of HG_{f_α} is provable in $\mathbf{PA}$ for all $\alpha < \varepsilon_0$ but it is unprovable for $\alpha = \varepsilon_0$.

Let EWD_g, where g denotes a natural function, be the modification of EWD in which $\mathsf{next}(w, m) := r * s * \cdots * s$ with $g(m+1)$ copies of s, $*$ denoting concatenation, for the case $f(n) > 0$ in the definition of the Worm Principle.

By the observations above, if one can make sure that Weiermann's result for the standard Hydra Game also applies to the slightly modified version and to the corresponding Hardy Hierarchy, then Weiermann's fine tuning for the Hydra Game also applies to the Worm Principle. It is very likely in the author's opinion that the additional "1" in the fundamental sequences does not harm Weiermann's analysis (see for example [13]), so that one can safely state the following [1].

Conjecture The fine tuning for the Worm Principle is the same as the fine tuning for the Hydra Game, i.e.

$$\mathbf{PA} \vdash \mathsf{EWD}_{|i|_{H_\alpha^{-1}(i)}} \Leftrightarrow \alpha < \varepsilon_0.$$

[1] A version of Conjecture 1 is settled in the positive by G. Lee and A. Weiermann in [10].

3 Worms and Gaps

We show that reduction sequences of worms correspond to bad sequences of finite sequences of natural numbers with respect to Schütte-Simpson's notion of gap- embeddability (see [11]), which is the one-dimensional version of Friedman's notion of gap-embedding of finite trees. We infer some Friedman-style independence results. We also present a characterization of Schütte-Simpson's strong gap-embeddability (see [11]) in terms of provability logic due to Lev Beklemishev.

Schütte-Simpson gap-embeddability and Worms The main result of this section is a simple combinatorial proof showing that the reduction sequences determined by the Worm Principle are bad with respect to the one-dimensional version of Friedman's gap-embeddability relation, investigated in [11].

Definition 2. Given a natural number n, let S_{n+1} be the set of all finite sequences of natural numbers $< n+1$. If $s_1 = (a_0, \dots, a_k)$ and $s_2 = (b_0, \dots b_m)$ are in S_{n+1}, then a strictly monotone function $f : \{0, \dots, k\} \to \{0, \dots, m\}$ is an *embedding* of s_1 into s_2 if the following holds:

1. $a_i = b_{f(i)}$ for all $i < k$,
2. if $f(i) < j < f(i+1)$ then $b_j \geq b_{f(i+1)}$ for all $i < k, j < m$.

For $s_1, s_2 \in S_{n+1}$ we say that s_1 is *gap-embeddable* in s_2 iff there exists a gap-embedding f of s_1 into s_2, and denote this fact by $s_1 \leq_{ge} s_2$.

It is known from [11] that, for each natural number n, $\leq_{ge}$ is a well-quasi-order on the set S_{n+1} of all finite sequences of natural numbers $< n+1$.

Theorem 3. *Let $s_1, s_2 \in S_{n+1}$. Then*

$$s_1 \hookrightarrow_W s_2 \Rightarrow s_1 \not\leq_{ge} s_2$$

Proof. Let $s_1 = (a_0, \dots, a_k)$ and $s_2 = (b_0, \dots b_m)$ be s.t. $s_2 = \mathsf{next}(s_1, p)$ By the definition of the game, there are two cases:

Case 1. $a_k = 0$. Then $m = k-1$ and $s_2 = a_0 \dots a_{k-1}$. Obviously s_1 is not gap-embeddable in s_2.

Case 2. Let $i_0 = \max(\{j < k : a_j < a_k\})$. Let I_0 be $a_{i_0+1} \dots a_{k-1} a_k - 1$. (If i_0 is undefined then $I_0 = a_0 \dots a_{k-1} a_k - 1$). Then

$$s_2 = a_0 \dots a_{i_0} \underbrace{a_{i_0+1} \dots a_{k-1} a_k - 1 \dots a_{i_0+1} \dots a_{k-1} a_k - 1}_{(p+1) \text{ copies}} = a_0 \dots a_{i_0} I_0 \times (p+1).$$

Suppose *per absurdum* that there is an embedding f of s_1 into s_2.

First consider the case that $i_0 = k-1$. Then $s_2 = a_0 \dots a_{k-1}(a_k - 1) \times p + 1$. It is easily seen that there cannot be an embedding of $s_1 = a_0 \dots a_{k-1} a_k$ in s_2. The cases $k = 0$ and $p = 0$ are similarly trivial. We can therefore suppose that $k > 0, p > 0, i_0 \neq k - 1$.

Suppose then that $i_0 < k - 1$ or i_0 undefined. Let $f(k) = j_1$. Since f is strictly monotonic, $b_{f(k)} = b_{j_1}$ lies in s_2 above at least one occurrence of $a_k - 1$. Let $b_{j_0} = a_k - 1$ be the first such occurrence with $j_0 < j_1$. Let j be the minimum element in $]j_0, \ldots, j_1]$ s.t. there exists an $m \in [i_0 + 1, \ldots, k]$ s.t. $f(m) = j$. At worst we have $j = j_1$. We observe that necessarily it is the case that $m > i_0 + 1$. Otherwise all the images of the elements $a_{i_0+1}, \ldots, a_{k-1}, a_k$ would lie in s_2 between two successive occurrences of $a_k - 1$, but between two such occurrences in s_2 only the elements $a_{i_0+1}, \ldots, a_{k-1}$ occur. Therefore $m - 1 \in [i_0 + 1, \ldots, k]$, and, by choice of j, $f(m - 1) < j_0$. Then we have $f(m-1) < j_0 < f(m)$ and $b_{j_0} = a_k - 1 < b_{f(m)} \geq a_k$. This violates the gap condition. □

Strong gap-embeddability and Provability Logic After seeing Theorem 3, Lev Beklemishev obtained an interesting characterization of Schütte-Simpson *strong* gap-embeddability in terms of provability logics. *Strong* gap-embeddability, denoted by $\leq_{sge}$, is just $\leq_{ge}$ with the following additional *root condition*:

$$(\forall j)[j < f(1) \Rightarrow b_j \geq b_{f(1)}].$$

Strong gap-embeddability is introduced as a key auxiliary notion in [11]. Obviously $\leq_{sge} \subseteq \leq_{ge}$.

The relation $\geq_{sge}$ turns out to be equivalent to implication in a particular system of provability logic $\mathbf{GLP}^-$, introduced by Ignatiev in [8][2]. $\mathbf{GLP}^-$ is a fragment of Japaridze logic $\mathbf{GLP}$, axiomatized by the following axioms.

1. For each operator $[n]$ the following axioms[3]:
   (a) Boolean tautologies,
   (b) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$,
   (c) $[n]([n]\varphi \to \varphi) \to [n]\varphi$.
2. $[m]\varphi \to [n][m]\varphi$, for $m \leq n$,
3. $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$,

plus the rules of Modus Ponens and Necessitation.

We now present Beklemishev's result with his kind permission. Recall that worms correspond to closed polymodal formulas, i.e. formulas generated by $\top, \langle 0\rangle, \langle 1\rangle, \ldots, \langle n\rangle, \ldots$ alone, so that we can speak of provability of formulas involving worms in a system like $\mathbf{GLP}^-$.

Theorem 4 (Beklemishev [3]). *For every worm α, β the following holds.*

$$\alpha \leq_{ge} \beta \Leftrightarrow \mathbf{GLP}^- \vdash \beta \to \alpha.$$

[2]In Ignatiev's paper the logic is called $\mathbf{LN}$, since it depends on a parameter N. Beklemishev suggested to call $\mathbf{GLP}^-$ the same logic with no parameter.

[3]These are the axioms of Gödel-Löb logic $\mathbf{GL}$, see [4].

Proof. ($\Rightarrow$) Assume $\alpha \leq_{ge} \beta$. We prove $\mathbf{GLP}^- \vdash \beta \to \alpha$ by induction on the length of α. The statement is trivial, if α is empty.

If $\alpha = n\gamma$, then $\beta = \beta_0 n \beta_1$ with $\beta_0 \in S_n$ by the root condition and $\gamma \leq_{ge} \beta_1$. We have by the induction hypothesis $\mathbf{GLP}^- \vdash \beta_1 \to \gamma$, and hence $\mathbf{GLP}^- \vdash n\beta_1 \to n\gamma$. Therefore,

$$\begin{aligned} \mathbf{GLP}^- \vdash \beta_0 n \beta_1 \ &\to n\beta_1 \\ &\to_{I.H.} n\gamma = \alpha. \end{aligned}$$

This proves the claim.

($\Leftarrow$) Assume $\alpha \not\leq_{ge} \beta$. We prove $\mathbf{GLP}^- \nvdash \beta \to \alpha$ by constructing a Kripke model for $\mathbf{GLP}^-$ in which $\beta \to \alpha$ fails. Let $\beta = (b_1, \dots, b_k)$.

Recall from [8] that a Kripke model for $\mathbf{GLP}^-$ is a finite set equipped with strict partial orderings R_i for each $i \geq 0$ and satisfying the following condition (*):

$$xR_iy \text{ and } j < i \Rightarrow (xR_jz \Leftrightarrow yR_jz).$$

Consider a model $\mathcal{K}$ whose nodes are labelled by $\{0, \dots, k\}$ and there is an arrow R_j between n_i and n_{i+1} exactly when $b_{i+1} = j$. This may not yet be an $\mathbf{GLP}^-$-model, because of the failure of transitivity or condition (*) in $\mathcal{K}$. However, there is a minimal $\mathbf{GLP}^-$-model $\mathcal{K}'$ containing $\mathcal{K}$, which is called the $\mathbf{GLP}^-$-closure of $\mathcal{K}$ [8]. It is obtained from $\mathcal{K}$ by adding appropriate arrows implied by transitivity and (*). We set xQ_ny if xR_iy for $i \geq n$ or yR_ix for $i > n$, and xS_ny if xR_ny or $\exists s \exists x_1 \dots \exists x_s$ such that $xQ_nx_1Q_nx_2 \dots Q_nx_sR_ny$. The $\mathbf{GLP}^-$-closure of $\mathcal{K}$ is the structure $\langle K, S_0, S_1, \dots, \rangle$ (see [8], Proposition 3.9, and observe that in $\mathcal{K}$ only the arrows R_j for $j \in \{0, \dots k\}$ are relevant).

It is obvious that at the node 0 of the model $\mathcal{K}$ (and in $\mathcal{K}^*$) the formula is true. It remains to prove that $\mathcal{K}^*, 0 \nVdash \alpha$.

Assume $\mathcal{K}^*, 0 \Vdash \alpha$. Then there is a sequence of nodes $0 = x_0, x_1, \dots, x_m \in \mathcal{K}$ such that $x_i S_{a_{i+1}} x_{i+1}$, for each $i < m$, where $\alpha = (\alpha_1, \dots, \alpha_m)$. We claim that this sequence defines a gap-embedding of α into β.

To this end we analyze the structure of $\mathcal{K}^*$. We claim that $\mathcal{K}^*$ satisfies the following property (**): if iS_qj (in $\mathcal{K}^*$), then (a) $i < j, (b) (j-1)R_qj$ (in $\mathcal{K}$), and (c) for every s such that $i \leq s < j$ there holds $b_{s+1} \geq q$. This property can be easily verified by induction on the generation of the $\mathbf{GLP}^-$-closure. It is clearly satisfied for $\mathcal{K}$: iR_qj in $\mathcal{K}$ implies by definition that $b_j = j$ and $i = j-1$, so the property holds. Now suppose that iS_qj in $\mathcal{K}^*$. Then either iR_qj (in $\mathcal{K}$) or there exists a chain such that $iQ_qi_1Q_q \dots Q_qi_sR_qj$. In the first case, the property obviously holds. For the second case, we just consider, for brevity, the case of $s = 1$, so that we have $iQ_qi_1R_qj$. By definition of $\mathcal{K}$ we have $i_1 = j-1$, which verifies part (a) of property (**), and $b_j = q$, which implies $(j-1)R_qj$ and verifies part (b) of property (**). By definition of Q_q we have two cases: either (1) iR_ni_1 for some $n \geq q$ or (2) i_1R_ni for some $n > q$. Consider case (1). By definition of $\mathcal{K}$ we have $b_{i_1} = n \geq q$ and $i = i_1 - 1$, so that $i = j-2$. So we have $b_{i+1} = b_{i_1} = n \geq q$ and $b_{j-1+1} = q$, verifying part (c) of property (**). So the property is verified in case (1). We now claim that case (2) is impossible. Suppose otherwise. Then we have i_1R_ni for some $n > q$. Also, we have

$i_1 R_q j$. So, by condition (*), we also have $iR_n j$. By definition of the model $\mathcal{K}$ this implies that $b_j = n > q$, but we already know that $b_j = q$.

Now, it immediately follows from $x_i S_{a_{i+1}} x_{i+1}$ in $\mathcal{K}^*$ that $(x_{i+1} - 1) R_{a_{i+1}} x_{i+1}$ in $\mathcal{K}$ and hence $b_{x_{i+1}} = a_{i+1}$. Besides, $b_{s+1} \geq a_{i+1}$, for all s such that $x_i \leq s < x_{i+1}$. In other words, the gap conditions for the embedding $i \mapsto x_i$ are satisfied. □

Observe that Theorem 4 implies Theorem 3, with $\leq_{sge}$ replacing $\leq_{ge}$, by the following argument. We know that $\mathbf{GLP} \vdash w_n \to \Diamond w_m$, if $m > n$. So, $\mathbf{GLP}$ (let alone $\mathbf{GLP}^-$) cannot prove $w_m \to w_n$, otherwise one contradicts Löb's theorem. So, w_n is not gap-embeddable into w_m. Also, since $\leq_{sge}$ is antisymmetric, we observe that for no two syntactically different worms α, β, $\mathbf{GLP}^-$ proves $\alpha \leftrightarrow \beta$. This also implies that the Normal Form Theorem for worms proved in [1] is not provable in $\mathbf{GLP}^-$ alone.

Another immediate interesting corollary of Theorem 4 is the following Theorem, also by Beklemishev, presented here with his kind permission.

Theorem 5 (Beklemishev). *For each n, the extensions of $\mathbf{GLP}^-$ in the language with n modalities by a set of closed modal formulas corresponding to negations of worms is finitely axiomatizable.*

Proof. For each natural number n, the worms in S_{n+1} are well-quasi-ordered by implication: $\alpha \leq_{sge} \beta$ iff $\mathbf{GLP}^- \vdash \beta \to \alpha$, iff $\mathbf{GLP}^- \vdash (\neg\alpha) \to (\neg\beta)$. Hence, for any fixed n, the closure of any set of negations of worms in S_{n+1} under logical consequence over $\mathbf{GLP}^-$ has finitely many minimal elements. These constitute the required finite axiomatization. □

Observe that the argument is reversible: assuming that any set of negations of worms is finitely axiomatizable, we infer the well-quasi-orderedness of the $\leq_{sge}$ relation. Since the latter is unprovable in $\mathbf{ACA}_0$, Theorem 5 is unprovable in $\mathbf{ACA}_0$.

Unprovability results and fine tuning We now consider some unprovability results stemming from Theorem 3. We only deal with elementary consequences of the results of the present paper and do not aim at completeness in any sense.

Let $A(n)$ be the following principle from [11].

$$\forall m \exists k \forall s_0, \ldots, s_k \in S_{n+1}[(\forall i \leq k)[\mathsf{sum}(s_i) \leq m \cdot (i+1)] \Rightarrow (\exists i < j \leq k)[s_i \leq_{ge} s_j]],$$

where the norm $\mathsf{sum}(\cdot)$ is the sum of the elements of the word s_i. Schütte and Simpson showed in [11] that the sentence $\forall n A(n)$ is true but unprovable in Peano Arithmetic. Now observe that the n-th worm w_n in a reduction sequence of the Worm Principle starting with a worm w_0 satisfies $\mathsf{length}(w_n) \leq \mathsf{length}(w_0) \cdot (n+2)!$, where the norm $\mathsf{length}(\cdot)$ is the length function (see [2]). An immediate consequence of Theorem 3 is the unprovability of the following (refinable) version of the Schütte-Simpson principle.

$$\forall n B(n) \equiv \forall n \forall m \exists k \forall s_0, \ldots, s_k \in S_{n+1} \quad [(\forall i \leq k)[\mathsf{length}(s_i) \leq m \cdot (i+2)!] \Rightarrow (\exists i < j \leq k)[s_i \leq_{ge} s_j]].$$

Starting with a worm of length m, the rules of the Worm Principle determine a long bad sequence with respect to $\leq_{ge}$ and satisfying the norm condition. Therefore the Skolem function for $B(n)$ grows at least as fast as the termination function for the Worm Principle. But we known from [2] that the latter dominates every $\mathbf{PA}$-provably total function.

We now make some remarks on the fine tuning for our Schütte-Simpson-style principle $\forall n B(n)$.

From Conjecture 1 we have that $\mathbf{PA} \vdash \mathsf{EWD}_{|i|_{H_\alpha^{-1}(i)}}$ iff $\alpha < \varepsilon_0$. Observe that, analogously to above, the n-th worm w_n in a reduction sequence of EWD_g starting with a worm w_0 satisfies $\mathsf{length}(w_n) \leq \mathsf{length}(w_0) \cdot (g(n)+2)!$. So we have unprovability in $\mathbf{PA}$ of the following refinement of $\forall n B(n)$.

$$\begin{aligned}\forall n \forall m \exists k \forall s_0, \ldots, s_k \in S_{n+1} \quad & [(\forall i \leq k)[\mathsf{length}(s_i) \leq m \cdot (|i|_{H_{\varepsilon_0}^{-1}(i)} + 2)!] \\ & \Rightarrow (\exists i < j \leq k)[s_i \leq_{ge} s_j]].\end{aligned}$$

Gordeev's gap-embeddability and Worms In [6] Gordeev introduced a variant of gap-embeddability, called *symmetrical* gap-embeddability, in order to obtain mathematically neat independence results from strong systems of arithmetic. We wish to point out that we can get the analogue of Theorem 3 for Gordeev's notion of gap-embeddability, essentially by the same "combinatorial" proof.

Let us first recall Gordeev's variant of gap-embeddability.

Definition 3. If $s_1 = (a_0, \ldots, a_k)$ and $s_2 = (b_0, \ldots b_m)$ are in S_{n+1}, then a strictly monotone function $f : \{0, \ldots, k\} \to \{0, \ldots, m\}$ is a *symmetric embedding* of s_1 into s_2 if the following holds:

1. $a_i \leq b_{f(i)}$ for all $i \leq k$,
2. if $f(i) < j < f(i+1)$ then $b_j \geq \min(a_j, a_{j+1})$ for all $i < k, j < m$.

For $s_1, s_2 \in S_{n+1}$ we say that s_1 is *Gordeev-gap-embeddable* in s_2 iff there exists a symmetric embedding f of s_1 into s_2, and denote this fact by $s_1 \leq_{Gge} s_2$.

At the end of our proof of Theorem 3 we have, by the same reasoning, that $f(m-1) < j_0 < f(m)$. But we can show that $b_{j_0} < \min\{a_{m-1}, a_m\}$: $b_{j_0} = a_k - 1$ by choice, and $a_{m-1}, a_m \geq a_k$ since m (by definition) and $m-1$ (by proof) are in $[i_0+1, \ldots k]$, and i_0 equals by definition the maximum element smaller than k such that $a_j < a_k$. Hence we have the following.

$$s_1 \hookrightarrow_W s_2 \Rightarrow s_1 \not\leq_{Gge} s_2.$$

Hence one can get fine unprovability results for Gordeev's notion of gap-embeddability as in the previous paragraph.

On a final note we point out that, as observed by Beklemishev and the present author, no logic between $\mathbf{GLP}^-$ and $\mathbf{GLP}$ characterizes Gordeev's gap-embeddability in the sense of Theorem 4. This can be seen as follows. Suppose that the analogue of

Theorem 4 holds for some system $\mathbf{L}$ between $\mathbf{GLP}^-$ and $\mathbf{GLP}$. In particular we have, for any worms α, β, $\alpha \leq_{Gge} \beta \Rightarrow \mathbf{L} \vdash \beta \to \alpha$. Then the following inferences have to be admissible in the system, where α is a worm.

1. $\langle n \rangle \top \to \langle m \rangle \top$, for $m \leq n$.
2. $\langle n \rangle \langle m \rangle \alpha \to \langle n' \rangle \langle m \rangle \alpha$, for $n' \leq n$.

But then we can reason as follows. $\mathbf{GLP}^- \vdash \langle 1 \rangle \langle 0 \rangle \top \leftrightarrow \langle 1 \rangle \top \wedge \langle 0 \rangle \top$. By the inference 1 above we have $\mathbf{L} \vdash \langle 1 \rangle \top \wedge \langle 0 \rangle \top \leftrightarrow \langle 1 \rangle \top$. So $\mathbf{L} \vdash \langle 1 \rangle \langle 0 \rangle \top \leftrightarrow \langle 1 \rangle \top$. By the the inference 2 above we have $\mathbf{L} \vdash \langle 1 \rangle \langle 0 \rangle \top \to \langle 0 \rangle \langle 0 \rangle \top$, so that $\mathbf{L} \vdash \langle 1 \rangle \top \to \langle 0 \rangle \langle 0 \rangle \top$. But $00 \not\leq_{Gge} 1$, a contradiction.

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