Learning to Win Process-Control Games Watching Game-Masters *

John Case[†]
University of Delaware

Matthias Ott[‡]
Universität Karlsruhe

Arun Sharma[§]
University of New South Wales

Frank Stephan[¶]
Universität Heidelberg

Abstract

The present paper focuses on some interesting classes of *process-control games*, where *winning* essentially means successfully controlling the process. A *master* for one of these games is an agent who plays a winning-strategy. In this paper we investigate situations, in which even a complete model (given by a program) of a particular game does not provide enough information to synthesize — even incrementally — a winning strategy. However, if in addition to getting a program, a machine may also watch masters play winning strategies, then the machine is able to incrementally learn a winning strategy for the given game. Studied are successful learning from *arbitrary* masters and from pedagogically useful *selected* masters. It is shown that selected masters are strictly more helpful for learning than are arbitrary masters. Both for learning from arbitrary masters and for learning from selected masters, though, there are cases where one can learn programs for winning strategies from masters but not if one is required to learn a program for the master's strategy itself. Both for learning from arbitrary masters and for learning from selected masters, one can learn strictly more watching $m + 1$ masters than one can learn watching only m. Lastly a simulation result is presented where the presence of a selected master reduces the complexity from infinitely many semantic mind changes to finitely many syntactic ones.

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†Department of CIS, University of Delaware, Newark, DE 19716, USA, Email: case@cis.udel.edu

‡Institut für Logik, Komplexität und Deduktionssysteme, Universität Karlsruhe, 76128 Karlsruhe, Germany, Email: m_ott@ira.uka.de. Supported by the Deutsche Forschungsgemeinschaft (DFG) Graduiertenkolleg "Beherrschbarkeit komplexer Systeme" (GRK 209/2-96).

§School of Computer Science and Engineering, University of New South Wales, Sydney 2052, Australia, Email: arun@cse.unsw.edu.au. Supported by Australian Research Council Grant A49600456.

¶Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany, Email: fstephan@math.uni-heidelberg.de. Supported by the Deutsche Forschungsgemeinschaft (DFG) Grant Am 60/9-1.

1 Introduction

To learn to win games such as chess, besides exploring the game tree with many practice games, it is also useful, or may even be necessary, to study the games of master players.[1] We do not have much access to the masters' actual strategic programs mostly stored in their subconscious wetware. We have, instead, access to their game-playing *behavior*. It is also apparently useful to study the (game-playing) behavior of masters who play with very different styles. For example, it is likely better to study the behavior of *both* Kasparov and Deep Blue[2] than to study only one of them.

In machine learning, the *behavioral cloning* approach to process-control, surveyed in [1], involves using data from the behavior of master or expert human controllers, in order to make complex control learning problems feasible. For example, it has been used successfully to teach an autopilot to fly an aircraft simulator [1, 5, 16, 24, 25] and to teach a machine to operate efficiently a free-swinging shipyard crane [1, 29]. Behavioral cloning partly motivates the present paper.

For us the *masters* are players of *winning strategies* for the classes of process-control games described in Section 1.1 just below. Of course the experts behaviorally cloned in the machine learning experiments mentioned just above aren't necessarily playing exactly the same kinds of process-control games as we study herein, nor are they necessarily playing perfect, complete, winning strategies. Nonetheless, some of the parallels we describe, in the rest of this subsection, between these experimental machine learning results and our main theorems are very interesting and, we hope, instructive for the future.

In this paper we study situations in which the learnability of strategies *necessarily* depends on the fact that the learner, in addition to exploring a complete description of the game, may also watch the behaviour of master players. For pedagogical purposes, some masters may be better to watch than others. In [1, 5, 16, 24, 25] it is noted that better results were obtained using the data from some pilots rather than others. Theorem 2.6 in Section 2 below implies that some masters are strictly more helpful than others. Hence, we distinguish between whether we are using *arbitrary* or carefully *selected* masters.[3]

In [1, 5, 16, 24, 25] the learning program employed, C4.5 [22], did not merely learn to copy identically each pilot modeled. We show in Section 2, for both arbitrary (Theorem 2.3) and selected masters (Theorem 2.5), that there are cases where one can learn winning strategies for process-control games from masters *but not if one is required to copycat the master.* An interestingly contrasting theorem in the same section (Theorem 2.2) implies that, if a class of process-control games can be learned *incrementally*, i.e., after finitely many trial and error rounds, from *arbitrary* masters, then it can be incrementally learned by copycatting *selected* masters.

In the learning-to-fly project [1, 5, 16, 24, 25] it was discovered that C4.5 got confused if it received data from more than one pilot at a time. Seemingly contrasting with this, in Section 3 below, we show, for both arbitrary (Theorem 3.4) and selected masters (Theorem 3.7), for each

[1]In this paragraph, by *master players* we mean players who win, not players formally designated as Masters (as opposed to Grand Masters in chess, ...).

[2]In principle, in the case of Deep Blue, we could look at its actual strategic program, but, even Kasparov learned from watching Deep Blue's *behavior*.

[3]Formally, this distinction is handled definitionally by universal versus existential quantifiers over masters in positive assertions (see Definition 2.1 in Section 2 below).

$m \geq 1$, surprisingly, one can learn strictly more watching $m+1$ masters than one can learn watching only m.[4] Interestingly, the separation between learning from two and learning from one selected master(s) is witnessed by a class of games, which is essentially specified by the *natural* class of all trees that contain infinitely many infinite computable branches.

1.1 The Process-Control Games

In the present paper we focus on the learning of (programs for) winning strategies for two kinds of *process-control games*. The two kinds turn out to be, for all our purposes, mathematically *equivalent* [12]! The second kind is mathematically elegantly simple, so we state and prove our results in terms of it, but, although, this second kind *is* interesting in its own right, more of our motivation comes from the first kind. Again: all of our results straightforwardly carry over *mutatis mutandis* to the first kind of process-control game.

The process-control games of the first kind are called *closed computable games*. These games nicely model *reactive* process-control problems. The second are the *one-player immortality games* (synonymously: *branch games*). We describe each in turn, the first informally (with references) and the second in more detail.

To explain closed computable games, we show how to model an *archetypal* process-control problem as a closed computable game. Suppose we wish to keep the temperature t in a particular room between $t_{\min} = 18\,^\circ\mathrm{C}$ and $t_{\max} = 22\,^\circ\mathrm{C}$, inclusive, where the initial temperature is $t_0 = 20\,^\circ\mathrm{C}$. A temperature controller, which can sense the temperature in the room, and an *unseen* physical disturbance each act at discrete times $n = 0, 1, 2, \ldots$ on the temperature of the room as follows. At time n, the controller and the disturbance can and do choose respective actions a_n and d_n each in $\{-1, 0, 1\}$, where the resultant temperature, in degrees Celsius, at time $n+1$, is given by $t_{n+1} = f(t_n, a_n, d_n) = t_n + a_n + d_n$. The controller sees the temperature, not the disturbance, and, from its perspective, the temperature behaves indeterministically, yet, the *controller has to do well against all possible behaviors of the temperature and disturbance.* Equivalently, the controller needs a winning strategy for the associated two-player closed computable game we describe next. Player I is the controller and Player II is the *temperature*. Of course *we* know that Player II is a mere puppet of Player I *and the unseen disturbance*, but Player I can *see* the temperature, so it is better to model Player II as the temperature. A *play* of the game is just an alternating infinite sequence $a_0 t_1 a_1 t_2 \ldots$ of controller actions and temperatures, i.e., of moves of Players I and II. Player I *wins* the play $a_0 t_1 a_1 t_2 \ldots$ iff $(\forall n)[t_{\min} \leq t_n \leq t_{\max}]$. The *goal set* for Player I is (by definition) the set of all plays $a_0 t_1 a_1 t_2 \ldots$ where Player I wins. In topology, *closed sets* (by definition) contain their limit points. The game we have described is called *closed* since the goal set for Player I is a closed set. I.e., *if* every finite initial segment $\sigma_n = a_0 t_1 a_1 \ldots a_{n-1} t_n$ of an infinite play yields no loss for the controller (i.e., if, for each $n \geq 1$, the temperature is between $t_{\min}$ and $t_{\max}$ at times $m = 1, \ldots, n$), *then* the *limit point* $a_0 t_1 a_1 \ldots$ of the sequence $(\sigma_n)_{n \in \omega}$ is a win for the controller, i.e., is in the goal set for Player I.[5] An example *winning strategy* for Player I, the controller, is as follows.

$$a_n = \begin{cases} +1 & \text{if } t_n \leq 20 \text{ or } n = 0, \\ -1 & \text{if } t_n > 20 \text{ and } n > 0. \end{cases} \tag{1}$$

[4] In the project on teaching an autopilot, a separate attribute distinguishing one pilot from another was not used; hence, this may partly explain the contrast.

[5] For more formal treatment, see [8, 12, 14, 28].

We have defined the winning strategy (1) by an informal algorithm, or program; hence, it's clearly computable. A human *master* playing strategy (1) would have stored in her head this or an equivalent program. Formally, the watchable *behavior* would be an enumeration of the pairs (t, a) such that t is a temperature that could be observed and a is this master's response.

Next we describe the mathematically equivalent *one player immortality games*. As an informal example, consider a robot which is placed in a (finite or infinite) environment. The robot's job is to keep exploring its environment yet not get trapped or destroyed. To help it, it has a model of its environment, from which it can generate, for example, a map showing the dangerous spots. If we model finite environments as deterministic finite automata [23], then, in these cases, the one-player immortality game can be modeled as follows. Given a finite automaton, a winning strategy is an infinite word such that the finite automaton never visits a rejecting state when run on this word.

Formally, and in general, a *one player immortality game* is (by definition) a computable tree containing at least one infinite (computable) branch. The player starts at the root, and its moves must take it successively further from the root. The winning strategies are exactly the infinite branches of the trees. The conventional, master-free, strategy learning scenario is: given an enumeration of the graph, or even a program, of the game tree, incrementally synthesize a program for following *some* such winning strategy, i.e., for traveling along some *infinite* branch. Death or entrapment is modeled, then, by the player getting stuck on a *finite* branch.

1.2 The Power of Watching Masters

As shown in [12], there are classes $\mathcal{C}$ of immortality games such that no machine can synthesize a winning strategy for every game $G \in \mathcal{C}$ *in the limit*, given an enumeration of the graph, or even a program, of G as input. However, it is reasonable that one can overcome such limitations by presenting to the machine an enumeration of a winning strategy as additional input, that is, the learner may *watch a master*. In this work we study the power of (several variants of) this new learning notion. It is *important* to note that for all our results in Sections 2 and 3, where we compare different models of learning from masters, it *does not matter* whether the game tree is presented by a program, or by an enumeration of its graph! On the other hand, this matters when one compares master learning to conventional strategy learning. This comparison is the topic of Section 4, and there, we will also discuss the effect of the two different input models on this comparison.

In order to demonstrate the additional power, which a learner may gain from watching a master, we consider the following illustrative example [20]. Let the tree T_e consist of branches $f_k = ka_0a_1\ldots$ for every natural number $k \in \omega$, that is, T_e has an infinitely branching root such that every successor k of the root is only extended by the branch f_k. The branches f_k may be finite or infinite, depending on the e-th recursively enumerable set W_e (of an standard enumeration of all r.e. sets). More precisely, we let $0a_0\ldots a_n$ be "on the tree" T_e iff a_m is the smallest number such that $|W_{e,a_m}| \geq m$, for $m = 0,\ldots,n$. Here, $(W_{e,n})_{n\in\omega}$ denotes a finite approximation of W_e. For $k > 0$, $k0^n$ is on the tree, iff $|W_{e,n}| < k$. It is not difficult to see, that the branch f_0 is infinite iff W_e is infinite, and that, for $k > 0$, f_k is infinite iff $|W_e| < k$. Thus, every tree T_e has an infinite branch.

Assume now, that M is a machine, which computes, for all e, from a program of T_e a sequence $i_0i_1\ldots i_miii\ldots$ which stabilizes on a program i for an infinite branch $f = f_k$ of T_e.

Then it holds that W_e is infinite iff $k = 0$. Note that $k = f_k(0) = f(0)$ can be computed using the program i. Thus, we have a procedure which decides the index set $Inf = \{e\colon W_e \text{ infinite}\}$ in the limit, that is, Inf is Δ_1 with respect to the arithmetical hierarchy. However, as well known, Inf is Π_2-complete, which is a contradiction.

Thus, a conventional branch learning machine can not synthesize infinite branches for all T_e even if it gets a program of T_e as input (instead of just an enumeration of T_e).

Now, consider a learner, who gets a program j for T_e and who watches an enumeration $f(0)f(1)\ldots$ of an infinite branch of T_e. Clearly, having seen $f(0)$ the learner knows *that the branch extending* $k = f(0)$ *is infinite* and can then, using the program j of T_e, compute a program for the infinite branch f. This demonstrates that learning by watching masters may be extremely more powerful than conventional branch learning. It even allows the learner to find a branch *without* any mind changes. Moreover, the learner can even *identify* the input master, instead of learning some possibly different, infinite branch of T_e. In this work we analyze this additional power gained by watching masters and compare several interesting variants of this learning notion.

Note that we could easily code e into the beginning of the tree T_e. Then it is still impossible to synthesize strategies for this class in the limit, however, a learner who watches a master can still be successful on this class, *even* if the learner gets only an enumeration of the graph of T_e as input. By similar reasoning, all our results in Sections 2 and 3 are, as already noted, independent of the input model for the trees. Therefore, we will base our, now following, more formal treatment on the easier model, in which the learner gets a *program* for the game tree as input.

2 Learning from a Master

The natural numbers are denoted by ω. ω^* is the set of all *finite* sequences from ω, and ω^ω is the set of all *infinite* sequences from ω. We are using an acceptable programming system $\varphi_0, \varphi_1, \ldots$; the function computed by the e-th program within s steps is denoted by $\varphi_{e,s}$. *REC* is the set of all total computable functions. For strings $\sigma, \tau \in \omega^* \cup \omega^\omega$, $\sigma \preceq \tau$ means that σ is an initial segment of τ. $|a_1 \ldots a_n| = n$ denotes the length of a string $a_1 \ldots a_n \in \omega^*$. If $\sigma \in \omega^*$ and $\tau \in \omega^* \cup \omega^\omega$ then $\sigma\tau$ is the concatenation of the two strings. Let $\langle\cdot\rangle$ be a coding of ω^*, i.e., a bijective computable function $\langle\cdot\rangle\colon \omega^* \to \omega$, which is monotone with respect to subsequences:

$$(\forall \sigma, \tau \in \omega^*)[\sigma \text{ is a subsequence of } \tau \implies \langle\sigma\rangle \leq \langle\tau\rangle].$$

We identify finite strings with their code numbers. Total functions $f : \omega \to \omega$ are identified with the infinite string $f(0)f(1)\ldots$. We write $f[n]$ for the initial segment $f(0)\ldots f(n-1)$ of f.

$T \subseteq \omega^*$ is a *tree* if T is closed under initial segments. If $T \subseteq \{0,1\}^*$ then T is called a *binary tree*. We base our definitions on arbitrary trees, however, the theory remains the same if it is based on trees over a finite alphabet, e.g., on binary trees [20].

Elements of a tree are called *nodes*. If $A \subseteq \omega^* \cup \omega^\omega$ is a set of finite and infinite strings, then the prefix closure, $\{\sigma \in \omega^* : \sigma \preceq \alpha \text{ for some } \alpha \in A\}$, is a tree. We often will define trees by specifying only such a set A. A total function $f\colon \omega \to \omega$ is an infinite branch of T if $f[n] \in T$ for all $n \in \omega$. The *width* of a binary tree T is defined by $width(T) = \max_{n\in\omega} |T \cap \{0,1\}^n|$.

For background from inductive inference see, e.g., [18]. Remaining computability theoretic notation is from [17].

We are interested only in the class *Tree* of all computable trees which contain at least one infinite computable branch. If $f \in REC$ is an infinite computable branch of T we also say that f is *on* T. Moreover, in the context, when an f on T is given as input to a learner, such a branch is called a *master*.

In what follows, for convenience, we will say *branch* to refer only to infinite computable branches. Furthermore, also for convenience, we will sometimes speak of learning a branch when we mean learning a program for the branch.

Definition 2.1 A Turing machine[6] M learns a branch *from an arbitrary master* for a tree $T \in \mathit{Tree}$, if *for all* masters f on T and for all e with $\varphi_e = T$, the sequence $(M(e, f[n]))_{n\in\omega}$ *converges* to an infinite computable branch of T, i.e., there exists an i such that φ_i is an infinite branch on T and $M(e, f[n]) = i$ for almost all n.[7] For $\mathcal{C} \subseteq \mathit{Tree}$ we write $\mathcal{C} \in \mathbf{ArbMa}$ if there exists a Turing machine M which learns branches from an arbitrary master for every $T \in \mathcal{C}$.

A Turing machine M learns a branch *from a selected master* for a tree $T \in \mathit{Tree}$, if there *exists* a master f on T such that for all e with $\varphi_e = T$ the sequence $(M(e, f[n]))_{n\in\omega}$ converges to a (program for an) infinite computable branch of T. For $\mathcal{C} \subseteq \mathit{Tree}$ we write $\mathcal{C} \in \mathbf{SelectMa}$ if there exists a Turing machine M which learns branches from a selected master for every $T \in \mathcal{C}$.

If there exists an **ArbMa**- or **SelectMa**-learner which converges to a program *for the input master* (instead of just to a program for *any* infinite computable branch of T) for every tree T of a class $\mathcal{C}$, we say that $\mathcal{C}$ is learnable from arbitrary/selected masters *identically.* The corresponding classes are denoted by **ArbMaId** and **SelectMaId**.

The definitions directly imply **ArbMa** $\subseteq$ **SelectMa**, **ArbMaId** $\subseteq$ **ArbMa** and **SelectMaId** $\subseteq$ **SelectMa**. One can prove that these inclusions are proper. Thus, to identify a master is a proper restriction for both learning from arbitrary and learning from selected masters. This shows that the advantage in watching one master (rather than none) comes from ones creating ones own winning strategy, and not from being a copycat. This result is not as surprising for **ArbMa** since one can imagine masters who go out of their way to avoid being figured out. But for the selective version of master learning this result is much more interesting. It says that regardless of how skilled pedagogically is the selected master you are watching, if one can learn a winning strategy from it, then this is, in general, only possible by creating a new strategy which differs from that of the master.

The noninclusion **SelectMa** $\not\subseteq$ **ArbMa** shows that not all masters are equally helpful for a learner. We are even able to prove **SelectMaId** $\not\subseteq$ **ArbMa**. Thus, while watching some masters provides enough information to identify these masters, watching others may be absolutely useless. Surprisingly, the other direction of the inclusion, **ArbMa** $\subseteq$ **SelectMaId**, holds. I.e., if every master allows the learner to at least find some winning strategy, then there exists one master which can even be identified by the learner. In summary, this establishes the proper linear chain **ArbMaId** $\subset$ **ArbMa** $\subset$ **SelectMaId** $\subset$ **SelectMa**. Moreover, it holds that *Tree* $\notin$ **SelectMa**. I.e., even if a learner can watch "a most helpful" master and is only

[6]By a well known argument, we can assume, without loss of generality, that all learning machines considered in this paper are total [18].

[7]Note that the sequence $M(e, f[n])$ converges *syntactically* to a (program for an) infinite computable branch, i.e., our notion of learning corresponds to the version of learning in the limit, which is called **Ex**-style (or incremental) learning in the literature.

required to output any winning strategy, it is still not possible to learn such strategies for all games, which have one.

One complexity measure of a learning task is the number of mind changes[8] which a machine needs to stabilize on a program for the target object. With respect to this complexity measure, learning without any mind changes at all provides the strongest positive results one may obtain. Zero mind change learning is also called *finite* learning in the literature. All the separation results which we give in this section are established by classes of trees so that the positive half of the separation result is witnessed by a machine which makes *no* mind changes!

Theorem 2.2 **ArbMa** $\subseteq$ **SelectMaId**.

Proof. This simulation proof is related to the proof of Freivalds and Wiehagen that every computable function can be learned from an upper bound of any of its indices [9, 10].

For a finite set $I \subseteq \omega$ the *amalgamation* of I is the program $AM(I)$ which on input x searches per dovetailing for an $i \in I$ such that $\varphi_i(x)\downarrow$, and then outputs $\varphi_i(x)$, that is:

$$\varphi_{AM(I)} = \lambda x.\varphi_{(\mu\langle i,s\rangle[i\in I \wedge \varphi_{i,s}(x)\downarrow])_1}(x).$$

Let $\mathcal{C} \in$ **ArbMa** via M. For $T \in \mathcal{C}$ we set $mb(T) = \min\{i\colon \varphi_i \text{ is on } T\}$. The following machine $N(e, f[n])$ identifies the selected master $f = \varphi_{mb(T)}$ of T:

1. Input e, $f[n]$.
2. $h_n = M(e, f[n])$.
3. $I_n = \{i \leq h_n\colon \neg(\exists x < n)[\varphi_{i,n}(x)\downarrow \neq f(x)]\}$.
4. Output $AM(I_n)$.

Let e with $T = \varphi_e \in \mathcal{C}$ be given and select $f = \varphi_{mb(T)}$. Since M is an **ArbMa**-learner for $\mathcal{C}$ there exist j and n_0 such that φ_j is a branch of T and $h_n = M(e, f[n]) = j$ for all $n \geq n_0$. It follows $mb(T) \leq j$, in particular, $mb(T) \in I_n$ for all $n \geq n_0$, since $\varphi_{mb(T)} = f$. Furthermore, by definition of I_n there exists an $n_1 \geq n_0$ such that all φ_i with $i \in I_n$ are consistent with $f = \varphi_{mb(T)}$. Thus, for all $n \geq n_1$ we have $\varphi_{AM(I_n)} = f$.

Hence, $\mathcal{C}$ is in **SelectMaId** via N where the selected master is the one with the smallest index. □

Theorem 2.3 **ArbMa** $\not\subseteq$ **ArbMaId**. *Moreover, this noninclusion can be witnessed by a class of trees which is* **ArbMa***-learnable without any mind change.*

Proof. Consider the class $\mathcal{C} = \{T \in \mathit{Tree}\colon 0^\omega \text{ is on } T\}$. Clearly, $\mathcal{C}$ is finitely **ArbMa**-learnable via the machine which always outputs a program for 0^ω.

If $\mathcal{C}$ were in **ArbMaId** via M, then the machine $\lambda\sigma.M(e,\sigma)$ would **Ex**-identify REC where e is a program for the full tree ω^ω. But it is well known that REC is not **Ex**-identifiable [3]. □

[8] In the formal definition of *mind change* one allows the machine to output initially a special symbol "?" to indicate that it has yet not seen enough data to make up its mind for its first conjecture. So a mind change is said to happen, if $? \neq M(e, f[n]) \neq M(e, f[n+1])$.

Next we present a diagonalization which shows that the family *Tree* is not in **SelectMa**. This serves also as a preparation for Theorem 2.5, since the following diagonalization is used as a subconstruction there.

Theorem 2.4 *Tree* $\notin$ **SelectMa**.

Proof. For every $e \in \omega$ we construct trees T_e with $\sigma(0) = e$ for all $\sigma \in T_e$, $\sigma \neq \epsilon$. Each tree T_e contains exactly one infinite computable branch f. In the construction of each T_e we diagonalize against φ_e as a potential **SelectMa**-learner for T_e. We will diagonalize against φ_e by securing that φ_e makes enough "prediction errors" on f, that is, $\varphi_e(f[n])$ is undefined or does not equal $f(n)$ for infinitely many n. Since every **SelectMa**-learner M would yield a predictor $\lambda\sigma \in \omega^*.\varphi_{M(i,\sigma)}(|\sigma|)$ for f on $\varphi_i = T_e$, which is at most finitely often undefined or incorrect, this will suffice to diagonalize against all **SelectMa**-learners. T_e will be uniformly computable from e. Since e is encoded into the master, it is therefore not necessary to give an index of T_e as input to the diagonalized machine.

Formally, the tree T_e is constructed in stages. In each stage, we define strings σ_s and τ_s. The tree T_e is obtained by $T = \{\sigma_s \colon s \in \omega\}$. The strings τ_s point to the last node where we have diagonalized against φ_e:

Construction of $T_e = \{\sigma_s \colon s \in \omega\}$:

Stage 0:
$\sigma_0 = \tau_0 = e$.

Stage s+1:
If $(\exists\tau)[\tau_s \preceq \tau \preceq \sigma_s$ and $\varphi_{e,s}(\tau)\downarrow]$, then choose the smallest such τ and let

$$a = s + 1 + \varphi_e(\tau).$$
$$\sigma_{s+1} = \tau_{s+1} = \tau a. \tag{2}$$

Else let

$$a = s + 1.$$
$$\sigma_{s+1} = \sigma_s a,\ \tau_{s+1} = \tau_s. \tag{3}$$

End of Construction.

Note that $\{\tau_s \colon s \in \omega\} \subseteq \{\sigma_s \colon s \in \omega\} = T_e$.

The tree T_e is computable: Let $\sigma \in \omega^*$ be given. Clearly, if $|\sigma| = 1$ then $\sigma \in T_e$ iff $\sigma(0) = e$. If $|\sigma| \geq 2$ then let $t = \sigma(|\sigma| - 1)$. Now it is easy to see that σ is in T_e iff $\sigma \preceq \sigma_s$ for some $s \leq t$: Clearly, if $\sigma \notin T_e$ then $\sigma \not\preceq \sigma_s$ for any $s \leq t$. Otherwise, if $\sigma \in T_e$, then σ is a prefix of some string σ_s. Let s be the smallest stage such that $\sigma \preceq \sigma_s$. Then $\sigma_s \neq \sigma_{s-1}$ and, by construction, there exist $\tau \preceq \sigma_{s-1}$ and $a \geq s$ with $\sigma_s = \tau a$. Thus, $\sigma = \tau$, since otherwise $\sigma \preceq \tau \preceq \sigma_{s-1}$. But this implies $s \leq a = \sigma(|\sigma| - 1) = t$. In summary, this shows that T_e is computable.

We consider two cases in order to show that T_e contains exactly one infinite computable branch:

> *Case 1:* $\bigsqcup_{s \in \omega} \tau_s$ is infinite, that is, line (2) is executed infinitely often. Then $f = \bigsqcup_{s \in \omega} \tau_s$ is an infinite computable branch of T_e. It is also the only branch of T_e: Let g be a branch of T_e and n be a natural number. We choose a stage s such that

$|\tau_s| > n$. Then, after stage s, we only enumerate nodes σ into T_e with $\tau_s \preceq \sigma$. Thus, we get $g(n) = f(n)$.

Case 2: $\bigsqcup_{s\in\omega} \tau_s$ is finite, that is, line (2) is executed only finitely often. In this case there exists a finite set $F \subseteq \omega^*$, a string σ, and an infinite computable sequence of stages $s_0, s_1, \ldots$ such that $T_e = F \cup \{\sigma s_0 s_1 \ldots\}$. Thus, $\sigma s_0 s_1 \ldots$ is the only branch of T_e and is computable, too.

Assume now, that the class $\mathcal{C} = \{T_e : e \in \omega\}$ is in **SelectMa** via the machine M. We choose $h \in REC$ with $(\forall e)[\varphi_{h(e)} = T_e]$ and define g according to

$$g(\sigma) = \begin{cases} \varphi_{M(h(\sigma(0)),\sigma)}(|\sigma|) & \text{if } |\sigma| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since g is computable there exists an e with $g = \varphi_e$. Let f be the unique branch on T_e. Thus, M learns a branch for T_e from the master f. Since f is the only branch of T_e this means that there exist numbers i and m such that $\varphi_i = f$ and $M(h(e), f[n]) = i$ for all $n \geq m$, in particular, $g(f[n]) = \varphi_i(|f[n]|) = f(n)$ for all $n \geq m$.

Assume that line (2) is executed at most finitely often, that is, there exists a t such that line (2) is not executed in any stage $s \geq t$. Then $\tau_s = \tau_t$ for all $t \geq s$ and $f = \tau_t s_0 s_1 \ldots$ for an infinite computable sequence of stages $s_0, s_1, \ldots$. We choose a number k such that $|\tau_t s_0 \ldots s_k| \geq m$ and let $\tau = \tau_t s_0 \ldots s_k$. Then, $\varphi_e(\tau) = f(|\tau|)$, in particular, $\varphi_e(\tau)$ is defined. Thus, there exists an $s \geq t$ such that $\tau_s = \tau_t \preceq \tau \preceq \sigma_s$ and $\varphi_{e,s}(\tau)\downarrow$. But then line (2) is executed in stage $s+1$, which is a contradiction.

Hence, line (2) is executed infinitely often, that is, $f = \bigsqcup_{s\in\omega} \tau_s$. Then there exists an s with $|\tau_s| > m$. Without loss of generality, we assume that $\tau_s \succ \tau_{s-1}$. Let τ and a from stage s be given such that we set $\tau_s = \tau a$ in this stage. Then, $|\tau| \geq m$ so that $g(\tau) = f(|\tau|)$. This implies

$$\varphi_e(\tau) = g(\tau) = f(|\tau|) = \tau_s(|\tau|) = a = s + 1 + \varphi_e(\tau),$$

which is a contradiction. $\square$

Theorem 2.5 **SelectMa** $\not\subseteq$ **SelectMaId**. *Moreover, this noninclusion can be witnessed by a class of trees which is* **SelectMa**-*learnable without any mind change.*

Proof. As already indicated above, we will build trees T_e for $e \in \omega$ which use the construction from Theorem 2.4 as a subconstruction. Each tree T_e has the form $T_e = \bigcup_{i\in\omega} U_e^i$ such that $ei \preceq \sigma$ for all $\sigma \in U_e^i$, $|\sigma| \geq 2$, and each tree U_e^i contains at most one infinite computable branch f. In every tree U_e^i we will try to diagonalize against φ_e as a potential **SelectMaId**-learner for T_e, very similarly to Theorem 2.4. However, we organize this diagonalization in such a way that it will only succeed in U_e^0 and in all trees U_e^i, for $i > 0$, such that φ_i is an infinite branch of T_e. All other trees U_e^i will be finite. This implies that for $i > 0$ every tree U_e^i contains an infinite branch iff φ_i is an infinite branch of T_e. This idea is the key to achieve $\mathcal{C} \in$ **SelectMa**, since the branch f of such an infinite tree U_e^i with $i > 0$ fulfills $f(1) = i$, that is, $f(1)$ is a program for an infinite branch of T_e. Therefore, these branches can be used as selected masters.

Construction of $T_e = \{\sigma_s^i \colon i, s \in \omega\}$:

Stage 0:

For all $i \in \omega$: $\sigma_0^i = \tau_0^i = ei$, $x_0^i = 2$.

Stage s+1:

1. For all $i > 0$:

   If $(\forall x < x_s^i)[\varphi_{i,s}(x) \downarrow]$ and $(\exists t \leq s)(\exists j)[\varphi_i[x_s^i] \preceq \sigma_t^j]$ then let

   $$x_{s+1}^i = x_s^i + 1.$$

   Else let

   $$x_{s+1}^i = x_s^i.$$

   $\{$*Check whether φ_i seems to become a branch of T_e so that U_e^i may be extended in step 2.*$\}$

2. For all i:

   If $i = 0$ or $x_{s+1}^i > x_s^i$ then

   if $(\exists \tau_i)[\tau_s^i \preceq \tau_i \preceq \sigma_s^i$ and $\varphi_{e,s}(\tau_i) \downarrow]$, then choose the smallest such τ_i and let
   $a_i = s + 1 + \varphi_e(\tau_i)$,
   $\sigma_{s+1}^i = \tau_{s+1}^i = \tau_i a_i$.

   Else let
   $a_i = s + 1$,
   $\sigma_{s+1}^i = \sigma_s^i a_i$, $\tau_{s+1}^i = \tau_s^i$.

   Else let

   $$\sigma_{s+1}^i = \sigma_s^i,\ \tau_{s+1}^i = \tau_s^i. \tag{4}$$

End of Construction.

Very similarly to Theorem 2.4 one can show that each tree T_e is computable:

First note that for the construction of σ_{s+1}^i we just have to know finitely many nodes σ_t^j. Of course, we have to know σ_s^i. Apart from that, only the test whether $(\exists t \leq s)(\exists j)[\varphi_i[x_s^i] \preceq \sigma_t^j]$ in step 1 needs access to strings σ_t^j. But here, it suffices to check all σ_t^j with $j = \varphi_i(1)$ and $t \leq s$. Therefore, the above construction is effective, that is, we can effectively compute each string σ_s^i.

Let $\sigma \in \omega^*$ be given. Clearly, if $0 < |\sigma| \leq 2$ then $\sigma \in T_e$ iff $\sigma(0) = e$. If $|\sigma| \geq 3$ then let $i = \sigma(1)$ and $t = \sigma(|\sigma| - 1)$. Now it is easy to see that σ is in T_e iff $\sigma \preceq \sigma_s^i$ for some $s \leq t$: Clearly, if $\sigma \notin T_e$ then $\sigma \not\preceq \sigma_s^i$ for any $s \leq t$. Otherwise, if $\sigma \in T_e$, then σ can only be a prefix of some σ_s^i, since $j = \sigma_s^j(1)$ for all j. Let s be the smallest stage such that $\sigma \preceq \sigma_s^i$. Then $\sigma_s^i \neq \sigma_{s-1}^i$ and, by construction, there exist $\tau \preceq \sigma_{s-1}^i$ and $a \geq s$ with $\sigma_s^i = \tau a$. Thus, $\sigma = \tau$, since otherwise $\sigma \preceq \tau \preceq \sigma_{s-1}^i$. But this implies $s \leq a = \sigma(|\sigma| - 1) = t$. In summary, this shows that T_e is computable.

We now consider the trees $U_e^i = \{\sigma_s^i\colon\ s \in \omega\}$. If $i > 0$ and the sequence $(x_i^s)_{s\in\omega}$ is bounded then in step 2 almost always line (2) is executed. This implies that U_e^i is finite, in particular, U_e^i has no infinite branch.

For $i = 0$ and for all i such that x_i^s goes to infinity for $s \to \infty$, the *if*-branch of step 2 is executed infinitely often. Here, we essentially carry out the same construction as in Theorem 2.4. Therefore, very similarly to Theorem 2.4 one can show that in this case each tree U_e^i contains exactly one infinite computable branch.

Moreover, one proves similarly to Theorem 2.4 that the class $\mathcal{C} = \{T_e :\ e \in \omega\}$ is not in **SelectMaId**. We just recall some details to point out the few differences. Assume that M is a **SelectMaId**-learner for $\mathcal{C}$. We choose $h \in REC$ with $(\forall e)[\varphi_{h(e)} = T_e]$ and define g according to

$$g(\sigma) = \begin{cases} \varphi_{M(h(\sigma(0)),\sigma)}(|\sigma|) & \text{if } |\sigma| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is computable, and thus there exists e with $\varphi_e = g$. Now, let f be the master on T_e such that M identifies f, that is, there are numbers i and m such that $\varphi_i = f$ and $M(h(e), f[n]) = i$ for all $m \geq n$. Let $j = f(1)$. Then, f is an infinite branch of U_e^j. Therefore, we have either $j = 0$ or the sequence x_s^j goes to infinity for $s \to \infty$. Thus, the *if*-branch of step 2 is executed infinitely often. Now, one can complete the proof of $\mathcal{C} \notin$ **SelectMaId** analogously to Theorem 2.4.

Finally, we show that $\mathcal{C}$ is finitely **SelectMa**-learnable. By construction there exists an i such that φ_i is an infinite branch of U_e^0. Therefore, the sequence x_i^s goes to infinity for $s \to \infty$ which implies that U_e^i contains an infinite branch f_e. We select f_e as master. The **SelectMa**-machine for $\mathcal{C}$ waits until it has read $f_e[2]$ and then outputs $f_e(1) = i$, which is a program for a branch of T_e. □

Theorem 2.6 SelectMaId $\not\subseteq$ ArbMa. *Moreover, this noninclusion can be witnessed by a class of trees which is* **SelectMaId***-learnable without any mind change.*

Proof. Let $\mathcal{C} = \{\bigcup_{i\in\omega} iT\colon\ T \in \mathit{Tree}\}$. $\mathcal{C}$ is in **SelectMaId** via the machine M which outputs a program for $(f(0)\,\varphi_{f(0)})$ as soon as it has read $f(0)$ from the master f. Let $U = \bigcup_{i\in\omega} iT$ be a tree in $\mathcal{C}$ and φ_e be an infinite computable branch of T. Then M identifies the master $(e\,\varphi_e)$ of T without any mind change.

Assume that $\mathcal{C}$ were in **ArbMa** via some machine M. Choose an $h \in REC$ with $\varphi_{h(e)} = \bigcup_{i\in\omega} iT$ for all e and $T \in \mathit{Tree}$ such that $\varphi_e = T$. Then the machine N with $\varphi_{N(e,\sigma)} = \lambda x.\varphi_{M(h(e),0\sigma)}(x+1)$ learns branches from arbitrary masters for the class *Tree*. But *Tree* is not even in **SelectMa** by Theorem 2.4. □

3 Hierarchies for Learning from Several Masters

In the learning to fly project mentioned in the introduction Sammut et al. have used the data from *several* pilots to train their system [2, 26]. In this section we study the corresponding situation in our model, that is, we consider learning machines which can watch several input masters instead of just a single master. For both models — learning from several *arbitrary* masters, and learning from several *selected* masters — we prove that one can properly learn more from $m + 1$ masters than one can learn from m masters.

In the definition of learning from m masters we have to deal with the problem that a given tree may have less than m masters. This is connected with the question whether we would like to allow several copies of one and the same branch as input masters. It is very clear that for an **ArbMa**-learner several copies of one master would not be very helpful. Therefore, we require that the masters which are provided as input to an **ArbMa**-learner are different as long as this is possible. That is, if the learner requires m-input masters and

1. there are at least m distinct infinite branches on the input tree T, then the m input masters have to be pairwise distinct;

2. the number k of distinct infinite branches on the input tree T is less than m, then exactly k of the m masters have to be pairwise distinct.

However, for an **SelectMa**-learner such conditions are not necessary. Why should it, in some cases, not be pedagogically the best to present m-copies of one and the same master as input to an **SelectMa**-learner? But, of course, for an **SelectMa**-learner we also have the possibility to choose different masters if it is to the advantage of the learner.

In the following we write $icb(T)$ for the number of infinite computable branches of a tree $T \in \mathit{Tree}$. Note that T may have infinitely many infinite computable branches, in which case we write $icb(T) = \infty$.

Definition 3.1 A Turing machine M learns a branch *from arbitrary m masters* for a tree $T \in \mathit{Tree}$, if *for all* masters $f_1, \dots, f_m$ on T with

$$|\{f_1, \dots, f_m\}| \geq \min\{m, icb(T)\} \tag{5}$$

and for all e with $\varphi_e = T$ the sequence $(M(e, f_1[n], \dots, f_m[n]))_{n\in\omega}$ converges to a (program for an) infinite computable branch of T. M learns a branch *from selected m masters* for a tree $T \in \mathit{Tree}$, if there *exist* masters $f_1, \dots, f_m$ on T such that for all e with $\varphi_e = T$ the sequence $(M(e, f_1[n], \dots, f_m[n]))_{n\in\omega}$ converges to a (program for an) infinite computable branch of T.

The corresponding classes are denoted by $\mathbf{ArbMa}^m$ and $\mathbf{SelectMa}^m$.

The definition implies the following corollary:

Corollary 3.2 *For all $m \geq 1$: $\mathbf{ArbMa}^m \subseteq \mathbf{ArbMa}^{m+1}$ and $\mathbf{SelectMa}^m \subseteq \mathbf{SelectMa}^{m+1}$.*

In the following we show that the hierarchies from Corollary 3.2 are proper. We first establish the **ArbMa**-hierarchy and start here with the special case $m = 2$:

Theorem 3.3 $\mathbf{ArbMa}^1 \subset \mathbf{ArbMa}^2$*. Moreover, the noninclusion can be witnessed by a class of trees which is $\mathbf{ArbMa}^2$-learnable without any mind change.*

Proof. In the following, we write $W_{e,s}$ for $dom(\varphi_{e,s}) \cap \{0, \dots, s\}$.

We use a class of trees defined in [20]. These trees are based on a family of $0,1$-valued functions $\{\varphi_{g(i)}\}_{i\in\omega}$ ($g \in \mathit{REC}$) — constructed by Kummer and Stephan [13] — which satisfies the following conditions:

- $1^i0 \preceq \varphi_{g(i)}$,

- $\varphi_{g(i)}(x)$ is undefined for at most one x,
- if W_i is finite and φ_e is a total extension of $\varphi_{g(i)}$, then $e \geq |W_i|$.

For each i, we define the tree T_i according to

$$T_i := \{a_0 \dots a_n \colon 1^i 0 \preceq a_0 \dots a_n \wedge (\forall m \leq n)[\neg(\varphi_{g(i),n}(m)\downarrow \neq a_m)]\}.$$

Note that the only infinite computable branches of T_i are the total computable $0,1$-valued extensions of $\varphi_{g(i)}$, i.e., if $\varphi_{g(i)}$ is total, then $\varphi_{g(i)}$ is the only infinite computable branch of T_i, otherwise T_i has exactly two infinite computable branches.

The idea of the proof is that, if one gets two different masters of T_i as input, then one knows that $\varphi_{g(i)}$ is not total and moreover, one can compute the argument x on which $\varphi_{g(i)}$ is undefined (if f_1 and f_2 are the two input masters then x is the unique y with $f_1(y) \neq f_2(y)$). Now, one could achieve learnability *with at most one mind change* by outputting $\varphi_{g(i)}$ as long as the two input masters are equal. As soon as a y with $f_1(y) \neq f_2(y)$ occurs, the learner switches to the hypothesis

$$\lambda x. \begin{cases} 0 & \text{if } x = y, \\ \varphi_{g(i)}(x) & \text{otherwise.} \end{cases}$$

In order to achieve finite learnability, i.e., learnability without any mind change, we have to introduce a new branch in the tree T_i, which is infinite iff $\varphi_{g(i)}$ is total. This branch (called α_i) can be defined by

$$a_0 \dots a_n \in \alpha_i \iff (\forall m \leq n)[a_m = \mu s[\varphi_{g(i),s}(m)\downarrow]].$$

Now we set $U_i := 0\alpha_i \cup 1T_i$.

Every tree U_i has exactly two infinite computable branches. If $\varphi_{g(i)}$ is total, then these branches are $(0\,\alpha_i)$ and $(1\,\varphi_{g(i)})$. Otherwise, α_i is finite, but therefore the two extensions of $\varphi_{g(i)}$ are on T_i and thus, U_i also has two infinite computable branches. These cases are effectively distinguishable from the two input masters, by testing the first bits of f_1 and f_2! This implies that it is easy to finitely learn a branch from two masters. Thus for $\mathcal{C} = \{U_i : i \in \omega\}$ we get $\mathcal{C} \in \mathbf{ArbMa}^2$.

Now, let us prove that $\mathcal{C}$ is not in $\mathbf{ArbMa}^1$. Assume that $\mathcal{C}$ were in $\mathbf{ArbMa}^1$ as witnessed by some machine M. A set A is called *PA-complete* if every partial computable $0,1$-valued function has a total A-computable extension. Fix some set A which is low (i.e., $A' \equiv_T K$) and PA-complete [17]. It has been shown in [20] that from an enumeration of T_i one can enumerate an infinite computable branch $f(0)f(1)\dots$ of T_i relative to A. But an enumeration of T_i can be computed from i, thus from i we can also compute the enumeration of f relative to A.

Let $e(i)$ be an index of U_i with $e \in REC$. We feed the master $(1\,f)$ into the machine M, i.e., we compute the sequence $h(i,n) := M(e(i), 1f[n])$. Since M is an $\mathbf{ArbMa}^1$-learner for $\mathcal{C}$, this sequence will converge to a program for an infinite branch of U_i.

We will now show that $Inf = \{i : W_i \text{ infinite}\}$ is decidable in the limit relative to A. There are two cases:

Case 1: The sequence $h(i,n)$ converges to a program for the branch $(0\,\alpha_i)$. This implies that $\varphi_{g(i)}$ is total. Then $\varphi_{g(i)}$ is an infinite branch of T_i. Thus:

$$W_i \text{ finite} \iff |W_{i,s}| \leq g(i).$$

Case 2: The sequence $h(i,n)$ converges to a program j for an infinite branch $(1\,\beta)$ of U_i (i.e., a branch through T_i). To extract an infinite branch for T_i from $(1\,\beta)$ we let $\varphi_{u(k)} := \lambda x.\varphi_k(x+1)$. Then $u(j)$ computes an infinite branch of T_i. Now, we get

$$W_i \text{ finite } \iff |W_{i,s}| \leq u(h).$$

Both cases can be handled uniformly by defining $d(i,n) := \max\{u(h(i,n)), g(i)\}$. It follows that the following procedure N decides $\mathit{Inf} = \{i : W_i \text{ infinite}\}$ in the limit relative to A:

$$N(i,n) = \begin{cases} 0 & \text{if } |W_{i,n}| \leq d(i,n), \\ 1 & \text{if } |W_{i,n}| > d(i,n). \end{cases}$$

Since A is low, this implies $\mathit{Inf} \leq_T A' \equiv_T K$, which is a contradiction. □

Theorem 3.4 *For all $m \geq 1$: $\mathbf{ArbMa}^m \subset \mathbf{ArbMa}^{m+1}$. Moreover, the noninclusion can be witnessed by a class of trees which is $\mathbf{ArbMa}^{m+1}$-learnable without any mind change.*

Proof. Let $\mathcal{C}$ be the class in $(\mathbf{ArbMa}^2 - \mathbf{ArbMa}^1)$ from Lemma 3.3. We define:

$$\mathcal{C}' := \{0\,U_0 \cup 1\,U_1 \cup \ldots \cup (m-1)\,U_{m-1} : U_0, \ldots, U_{m-1} \in \mathcal{C}\}.$$

The class $\mathcal{C}'$ is in $\mathbf{ArbMa}^{m+1}$: Assume that M is a finite $\mathbf{ArbMa}^2$-learner for $\mathcal{C}$. Note that each tree $U \in \mathcal{C}$ contains at least two infinite branches so that each tree $T \in \mathcal{C}'$ contains at least $m+1$ pairwise different branches. Given T in $\mathcal{C}'$ and $m+1$ *pairwise different* masters of T (which exist), at least two of them are branches of one subtree $(i\,U_i)$ by the pigeonhole principle. It is easy to find two such input masters f_1 and f_2 in finite time by checking the first nodes of the masters. Moreover, from T one can uniformly compute an index e for U_i. Now, simulate the machine M on U_i and the masters $g_1 = (\lambda x.f_1(x+1))$ and $g_2 = (\lambda x.f_2(x+1))$, i.e., compute $h(n) = M(e, g_1[n], g_2[n])$. Since g_1 and g_2 are two different masters of U_i (note that $f_1(0) = f_2(0)$), the sequence $h(n)$ *finitely* converges to a program j for an infinite branch of U_i. So, if we output a program for $(i\,\varphi_j)$ as soon as $h(n)$ has converged to j, we finitely learn a program for an infinite branch of T.

But the class $\mathcal{C}'$ is not in $\mathbf{ArbMa}^m$: Assume $\mathcal{C}' \in \mathbf{ArbMa}^m$ as witnessed by M. We will show $\mathcal{C} \in \mathbf{ArbMa}^1$, which yields a contradiction. Assume that we get T in $\mathcal{C}$, and an enumeration $f(0)f(1)\ldots$ of a master f as input. From the index of T we can uniformly compute an index for $T' = 0\,T \cup 1\,T \cup \ldots \cup (m-1)\,T$, and the sequence $f(0)f(1)\ldots$ can effectively be translated into m sequences for $g_i = (i\,f)$ where $i = 0, \ldots, m-1$. Certainly, T' is in $\mathcal{C}'$, and $g_0, \ldots, g_{m-1}$ are m different masters of T'. We simulate M on T' and $g_0, \ldots, g_{m-1}$ by computing $h(n) = M(e, g_0[n], \ldots, g_{m-1}[n])$ where $\varphi_e = T'$. This sequence converges to a program for an infinite branch of T'. Thus, the sequence $(\lambda x.\varphi_{h(n)}(x+1))_{n\in\omega}$ converges to an infinite branch of T. *Contradiction.* □

The separation of $\mathbf{ArbMa}^{m+1}$ and $\mathbf{ArbMa}^m$ can also be witnessed by the natural class $\mathcal{C}_{m+1}$ of all computable binary trees from *Tree* which have at most $m+1$ (arbitrary) infinite branches. However, the $\mathbf{ArbMa}^{m+1}$ learner for $\mathcal{C}_{m+1}$ needs generally $\log(m)$ mind changes (instead of zero mind changes). Since such separation proofs, which use natural classes, are particularly interesting for inductive inference, we also present the corresponding proof here:

Theorem 3.5 *For all $m \geq 1$, the class $\mathcal{C}_{m+1}$ of all computable binary trees from Tree which have at most $m+1$ (arbitrary) infinite branches, is in* $\mathbf{ArbMa}^{m+1} - \mathbf{ArbMa}^{m}$.

Proof. We first contsruct a machine M which learns branches for $\mathcal{C}_{m+1}$ from arbitrary $m+1$ masters. Let $T \in \mathcal{C}_{m+1}$ be given. Note that, since T contains a most $m+1$ arbitrary infinite branches, ever infinite branch of T is computable. Let σ be a node of T such that T containst exactly one infinite (computable) branch extending σ (such a σ exists for every infinite branch of T). It is well known that from σ and an index e of T one can uniformly compute a program $b(e, \sigma)$ of the unique infinite branch on T extending σ (see, e.g., [6]). The machine M works as follows.

On input e and $f_1[n], \ldots, f_{m+1}[n]$ it first determines the set

$$A = \{f_i[n] \sqcap f_j[n] \colon 1 \leq i, j \leq n \text{ and } f_i[n] \neq f_j[n]\}$$

and chooses the lexicographical smallest minimal element σ from A, where the infimum $\sqcap$ and the minimal elements are taken with respect to the prefix order "$\preceq$". M then outputs the program $b(e, \sigma)$.

To prove the negative statement, we construct, for each (total) learner φ_e, a tree $T_e = \varphi_{g(e)}$ such that this tree diagonalizes against the learner φ_e. First we include the nodes $\sigma_k = 0^e 1^k 0^{n+2-k}$, for $k = 1, 2, \ldots, m$, into T_e. For every node σ_k the tree T_e will have at least one and at most two infinite branches extending σ_k. The nodes on T_e of length s are specified by the following algorithm:

> If $s \leq |\sigma_1|$ then T contains all nodes of length s which are prefixes of some σ_k with $k = 1, 2, \ldots, m$.
>
> Otherwise, $s > |\sigma_1|$ and, on level $s-1$, there are m nodes $\eta_k \succeq \sigma_k$, for $k = 1, \ldots, m$, plus perhaps one further node $\eta'_l \succeq \sigma_l$ for some l. Let $i = \varphi_e(g(e), \eta_1, \eta_2, \ldots, \eta_m)$, that is, i is the output of the learner φ_e on the initial segments $\eta_1, \eta_2, \ldots, \eta_m$ of m potential input masters. We now add the nodes on level s of T_e according to the following cases, where the first case, which applies, is chosen:
>
> 1. $\varphi_e(g(e), \eta_1 0, \eta_2 0, \ldots, \eta_m 0) \neq i$ or $\varphi_{i,s}$ is incomparable with all η_k, $1 \leq k \leq n$.
>   Then we add $\eta_1 0, \eta_2 0, \ldots, \eta_n 0$ to T_e.
> 2. The node η'_l exists and the previous case holds with η'_l in place of η_l.
>   Then we add $\eta'_l 0$ and all nodes $\eta_k 0$ with $k \neq l'$ to T_e.
> 3. The node $\varphi_{i,s}$ is a prefix of at least two nodes on level $s-1$.
>   Then every node η on level $s-1$ is extended to $\eta 0$ on level s.
> 4. The node $\varphi_{i,s}$ is comparable with a unique node η_k.
>   Then we add the nodes $\eta_1 0, \ldots, \eta_n 0$ plus $\eta_k 1$ to T_e.

Note that always one of the four cases in the algorithm holds. If $\varphi_{i,s}$ is incomparable to all nodes $\eta_1, \eta_2, \ldots, \eta_m$ then the first case applies, and if $\varphi_{i,s}$ is comparable to two of these nodes then the third case applies. Otherwise, the condition of the fourth case is valid. Furthermore, the second case may apply if η'_l exists.

In every of the four cases, only m or $m+1$ nodes are appended, thus, the tree has at most $n+1$ infinite branches and its width is at most $m+1$. On the other hand, in every stage we extend a node above each σ_k and so, T_e has m or $m+1$ infinite branches. Hence, T_e is in $\mathcal{C}_{m+1}$. Furthermore, there exists a smallest number a, $1 \leq a \leq 4$, such that the a-th case applied infinitely often during the construction of T_e.

If $a = 1$ then T_e has exactly m infinite branches $f_1, f_2, \ldots, f_m$, and, whenever the first case applies, we have $\sigma_k \preceq \eta_k \preceq f_k$, for $1 \leq k \leq n$. It follows that the learner φ_e, on the input masters $f_1, \ldots, f_m$, either makes infinitely many mind changes, or outputs infinitely often a hypothesis which is incomparable with all η_k and therefore different from all f_k. Thus, the learner does not $ArbMa^m$-learn a branch for T_e.

If $a = 2$ then the argumentation is analogous to the case $a = 1$, were η'_k is used instead of η_k.

If $a > 2$ then the first two cases applied only finitely often during the construction of T_e. We choose a sufficiently large number s_0 such that none of the cases 1 or 2 appears in any stage $s \geq s_0$. This, in particular, implies that φ_e does not make any more mind changes in stages $s \geq s_0$, that is, the learner φ_e has converged to a single hypothesis i. Note that case 4 is executed at most once after stage s_0: Assume that case 4 applies in stage $t \geq s_0$ and that, among others, the new nodes $\eta_k 0$ and $\eta_k 1$ are added to T_e in this stage. Now, case 4 is only applicable in a stage $s > t$ if the hypothesis φ_i eventually follows uniquely one of the new nodes $\eta_k 0$ or $\eta_k 1$. But then, one of the cases 1 or 2 applies first, which is a contradiction.

Thus, from some stage $s_1 \geq s_0$ on, only case 3 is executed which implies that the final hypothesis φ_i is not total. This completes the proof that φ_e is not an $\mathbf{ArbMa}^m$ learner for T_e. Hence, the class $\{T_e \colon \varphi_e \text{ total}\}$ is a subset of $\mathcal{C}_{m+1}$ but is not in $\mathbf{ArbMa}^m$. □

In order to prove the analogue to Theorem 3.4 for selective master learning, we use a *team* learning result from [6]. The term *team* in the context of learning was introduced by Smith [27]. Basically, a *team of size* m is a collection of m learning machines which simultaneously and independently try to solve a learning problem. A team of m machines is *successful*, if at least one of them is (and we may not know which one). Pitt [21] showed the surprising result that one out of m explanatory team function identification is equivalent to *probabilistic* explanatory identification (by a single machine) achieving probability of success $\frac{1}{m}$.

In [6] the identification of computable functions has been investigated when a tree is given as additional input to the learner, which contains the function as an infinite computable branch. In particular, [6] studies the team size which is necessary to identify all infinite branches of some naturally defined tree classes.

We say that a class $\mathcal{C} \subseteq$ *Tree* is in $[1,m]\mathbf{SelectMa}$, if there are m learning machines such that for each $T \in \mathcal{C}$ one of the m learning machines learns a branch for T from *one* selected master. Note that each machine of the team gets the *same* selected master of T as input. Using a result from [6] one can separate $[1,m+1]\mathbf{SelectMa}$ from $[1,m]\mathbf{SelectMa}$ as follows.

Theorem 3.6 *For all $m \geq 1$: $[1,m+1]\mathbf{SelectMa} \not\subseteq [1,m]\mathbf{SelectMa}$. Moreover, this noninclusion is witnessed by a class of trees such that each tree contains exactly one infinite computable branch.*

Proof. Let $M_1, \ldots, M_n$ be a team of M learning machines with two input arguments. In [6, Theorem 5.6.b] a *recursively enumerable binary* tree $T(M_1, \ldots, M_n)$ is constructed satisfying the following properties:

- *width*$(T(M_1, \ldots, M_n)) \leq n+1$,

- $T(M_1, \ldots, M_n)$ contains exactly one infinite computable branch f,
- for all $i = 1, \ldots, n$ and all e with $W_e = T(M_1, \ldots, M_n)$ it holds that the sequence $M_i(e, f[n])$ does not **Bc**-converge to f.

We consider the class

$$\mathcal{C} = \{T(M_1, \ldots, M_n)\colon\ M_1, \ldots, M_n \text{ is a team of } n \text{ learning machines}\}.$$

It follows that there exists no team $M_1, \ldots, M_n$ which **Bc**-learns the branch on every $T \in \mathcal{C}$, when a recursively enumerable index is given as additional input. Moreover, since each tree in $\mathcal{C}$ has at most width $n+1$, [6, Theorem 5.6.a] implies that there exists a team $M_1, \ldots, M_{n+1}$ which **Ex**-learns the branch on every $T \in \mathcal{C}$, when a recursively enumerable index is given as additional input, that is,

$$\begin{aligned}&(\forall T\in \mathcal{C})(\forall e)(\exists i, 1 \leq i \leq n+1)\\&\qquad [W_e = T \implies (M_i(e, f[n]))_{n\in\omega} \ \mathbf{Ex}\text{-converges to } f].\end{aligned}$$

As already indicated in [6] the class $\mathcal{C}$ of recursively enumerable binary trees can be effectively transformed into a class of *computable* trees (over alphabet ω) which are *equivalent* with respect to the above learnability properties. This can be seen, for example, by a transformation given in [11, Theorem 5]. For each e such that W_e is a recursively enumerable binary tree we define an tree $T^{rec}(e)$ according to

$$T^{rec}(e) = \{a_0 b_0 \ldots a_n b_n\colon\ (\forall m \leq n)[b_m = \mu t[a_0 \ldots a_m \in W_{e,t}]\}.$$

Clearly, $T^{rec}(e)$ is computable. It is also easy to see that there exists a $1:1$ degree preserving correspondence between the infinite branches of W_e and of $T^{rec}(e)$. The trees W_e and $T^{rec}(e)$ can be uniformly computed from each other, that is, from e one can compute an i with $\varphi_i = T^{rec}(e)$, and from i with $\varphi_i = T^{rec}(e)$ one can compute a j such that $W_j = W_e$. Moreover, the branches can be effectively computed from each other (even if we do not have a program for them, that is, if we can only access their values) assuming that in the one direction the recursively enumerable index e of the tree W_e is given as additional information. Thus, by reducing learning procedures for $\mathcal{C}$ and $\mathcal{C}^{rec}$ to each other, one sees that each successful team for $\mathcal{C}$ yields a successful team for $\mathcal{C}^{rec} = \{T^{rec}(e)\colon\ W_e \in \mathcal{C}\}$, and vice versa.

In summary, this implies that the class $\mathcal{C}^{rec}$ is a class of *computable* trees satisfying the following properties:

- Each tree $T \in \mathcal{C}^{rec}$ contains exactly one infinite computable branch f_T,
- $\neg(\exists M_1, \ldots, M_n)(\forall T \in \mathcal{C})(\forall e)(\exists i, 1 \leq i \leq n)$
  $[\varphi_e = T \implies (M_i(e, f[n]))_{n\in\omega}\ \mathbf{Bc}\text{-converges to } f_T]$,
- $(\exists M_1, \ldots, M_{n+1})(\forall T \in \mathcal{C})(\forall e)(\exists i, 1 \leq i \leq n+1)$
  $[\varphi_e = T \implies (M_i(e, f[n]))_{n\in\omega}\ \mathbf{Ex}\text{-converges to } f_T]$.

Since each tree $T \in \mathcal{C}^{rec}$ contains exactly one infinite computable branch it follows that $\mathcal{C}^{rec}$ is in $[1, m+1]$**SelectMa** but not in $[1, m]$**SelectMa**. Thus, the class $\mathcal{C}^{rec}$ witnesses our claim. □

We will use Theorem 3.6 to show that the **SelectMa**^{m}-hierarchy from Corollary 3.2 is proper:

Theorem 3.7 *For all $m \geq 1$:* $\mathbf{SelectMa}^m \subset \mathbf{SelectMa}^{m+1}$.

Proof. Let an arbitrary $m \geq 1$ and a class $\mathcal{C}_{2^m}$ in $([1, 2^m + 1]\mathbf{SelectMa} - [1, 2^m]\mathbf{SelectMa})$ be given, such that each $T \in \mathcal{C}_{2^m}$ contains exactly one infinite computable branch. The class $\mathcal{C}_{2^m}$ exists by Theorem 3.6. For each $T \in \mathcal{C}_{2^m}$ we define new trees $U(T) = 0\,T \cup 1\,T$. The class $\mathcal{C} = \{U(T) : T \in \mathcal{C}_{2^m}\}$ is not in $\mathbf{SelectMa}^m$: Assume by way of contradiction that $\mathcal{C} \in \mathbf{SelectMa}^m$ via the machine M. Let $\beta_1, \ldots, \beta_{2^m}$ be an enumeration of all boolean functions over $\{1, \ldots, m\}$, and g be a computable function such that $\varphi_{g(e)} = U(T)$ for all e with $\varphi_e = T$. Then $\mathcal{C}_{2^m}$ is in $[1, 2^m]\mathbf{SelectMa}$ via the team of machines $M_1, \ldots, M_{2^m}$ where M_i works as follows for $i = 1, \ldots, 2^m$:

$$h_i(e, f[n]) = M(g(e), (\beta_i(1)\, f[n]), \ldots, (\beta_i(m)\, f[n])),$$
$$\varphi_{M_i(e, f[n])} = \lambda x.\varphi_{h_i(e, f[n])}(x + 1),$$

To see that one of the machines M_i learns a branch for an arbitrary $T \in \mathcal{C}_{2^m}$ let $f_1, \ldots, f_m$ be m masters such that M learns a branch for $U(T)$ from $f_1, \ldots, f_m$. Since T contains exactly one infinite computable branch f, each master f_j has the form $(f_j(0)\, f)$. Thus, the machine M_i with $\beta_i(k) = f_k(0)$, for $k = 1, \ldots, m$, simulates M on $f_1, \ldots, f_m$. As soon as M converges to (a program for an) infinite branch u of $U(T)$, M_i will output a program for $\lambda x.u(x + 1)$ which is an infinite branch of T. Since $\mathcal{C}_{2^m}$ is not in $[1, 2^m]\mathbf{SelectMa}$ by assumption we get a contradiction.

But $\mathcal{C}$ is in $\mathbf{SelectMa}^{m+1}$: Let $\mathcal{C}$ be in $[1, 2^m+1]\mathbf{SelectMa}$ via the machines $M_1, \ldots, M_{2^m+1}$ and let an arbitrary tree $U(T) \in \mathcal{C}$ be given. Since $2^{m+1} \geq 2^m + 1$ we can use the first nodes of the $m + 1$ masters $f_1, \ldots, f_{m+1}$ to code the number i of the machine M_i which correctly infers the only infinite computable branch of T. Now, the learner can decode i from the input masters $f_1, \ldots, f_{m+1}$ and simulate M_i on, for instance, the master $\lambda x.f_1(x + 1)$ of T, which results in a sequence of hypotheses $(h_n)_{n \in \omega}$. Then the sequence $(0\, h_n)_{n \in \omega}$ will converge to (a program for) an infinite branch of $U(T)$ since $(h_n)_{n \in \omega}$ converges to (a program for) an infinite branch of T. $\square$

Theorem 3.7, in particular, implies that even our most powerful notion of master learning, namely, learning an arbitrary branch from several selected masters, is not strong enough to learn a branch for every tree:

Corollary 3.8 *For all $m \geq 1$: Tree* $\notin \mathbf{SelectMa}^m$.

We have established the $\mathbf{SelectMa}^m$-hierarchy by using a team learning result from [6]. However, interestingly, the separation of $\mathbf{SelectMa}^2$ and $\mathbf{SelectMa}^1$ is also witnessed by *natural* classes of trees, in particular, by the class *TreeInf* of all trees which contain infinitely many infinite computable branches. Again, because such seperations by natural classes are of particular interest, we explicitely state the corresponding proof.

Besides *TreeInf*, also the following natural classes witness $\mathbf{SelectMa}^2 \not\subseteq \mathbf{SelectMa}^1$ (see, e.g., [6] for definitions):

- the binary trees which contain only computable infinite branches,
- the binary trees of bounded width,

- the binary trees of bounded rank, and
- the binary trees of bounded variation.

Theorem 3.9 *The class TreeInf of all trees, which contain infinitely many infinite computable branches, is in* $\mathbf{SelectMa}^2 - \mathbf{SelectMa}^1$.

Proof. We first prove $\mathit{TreeInf} \in \mathbf{SelectMa}^2$. So, let an arbitrary tree $T \in \mathit{TreeInf}$ be given, and let e' be the smallest e such that φ_e is on T. Then there exists an x' such that for all $e < e'$, either $\varphi_e(x)\uparrow$ for some $x < x'$, or $\varphi_e[x'] \notin T$. Since T contains infinitely many branches, T contains an infinitely branching node, or, for every n, there exist two different branches f_1, f_2 on T with $f_1[n] = f_2[n]$. This implies that there exist an n' and two different branches f_1, f_2 on T with $f_1[n'] = f_2[n']$, $f_1(n') \neq f_2(n')$ and $n' + f_1(n') + f_2(n') > x'$. We choose these branches f_1 and f_2 as selected masters.

Now, the machine M witnessing $\mathit{TreeInf} \in \mathbf{SelectMa}^2$ works as follows. On input T, f_1, f_2 the machine M waits until it finds the first n with $f_1(n) \neq f_2(n)$. Then it computes $y = n + f_1(n) + f_2(n)$. Note that $n = n'$ and $y > x'$. From now on, M outputs in stage n the smallest $e \leq n$ such that

$$(\forall x < y)[\varphi_{e,n}(x)\downarrow] \text{ and } \varphi_e[y] \in T. \tag{6}$$

If no such e exists, then M outputs 0.

By definition, there exists an $n \geq e'$ such that $e = e'$ and n satisfy (6). On the other hand, no $e < e'$ can satisfy (6) for any n. Thus, from some point on, the machine will always output e' which is a program for a branch of T. Hence, M is a correct $\mathbf{SelectMa}^2$-learner for the class *TreeInf*.

We now prove the the negative statement. Assume by way of contradiction that the Turing machine M witnesses $\mathit{TreeInf} \in \mathbf{SelectMa}$. Let η_n denote the unique string with $n = \langle \eta_n \rangle$. We will construct a tree T such that M fails to $\mathbf{SelectMa}$-learn a branch for T as follows.

Construction of $T = \bigcup_{s\in\omega} T_s$:

Let e be an index of T (Recursion Theorem).

Stage 0: $T_0 = \emptyset$, $\mathit{queue}_0 = (\epsilon)$.

Stage s+1: Assume that $\mathit{queue}_s = (\sigma_0, \ldots, \sigma_q)$.
Compute $j = M(e, \sigma_0)$, where $\varphi_e = T$.
Check whether η_s satisfies the following conditions:
(a) $\sigma_0 \prec \eta_s$,
(b) $(\forall \tau, \sigma_0 \prec \tau \prec \eta_s)[\tau \in T_s]$,
(c) $(\exists x < |\sigma_0|)[\varphi_{j,s}(x)\uparrow]$ or $(\exists x < |\eta_s|)[\varphi_{j,s}(x)\downarrow \neq \eta_s(x)]$.

If η_s satisfies (a) – (c) then
let $T_{s+1} = T_s \cup \{\eta_s\}$;
if $M(e, \eta_s) \neq j$ then let $\mathit{queue}_{s+1} = (\sigma_1, \ldots, \sigma_q, \eta_s 0, \eta_s 1)$, (7)
else let $\mathit{queue}_{s+1} = \mathit{queue}_s$.
If η_s does not satisfy (a) – (c) then let $T_{s+1} = T_s$, $\mathit{queue}_{s+1} = \mathit{queue}_s$.

End of Construction.

Due to condition (b) every string η_s can only be included into T in stage $s+1$. Thus, T is clearly computable. Furthermore, note that for each string σ_0 there exist infinitely many $a \in \omega$ such that $\eta_s = \sigma_0 a$ satisfies conditions (a) — (c). Therefore, each string which is in some stage s head of $queue_s$ will eventually be extended.

In order to prove that T is actually in *TreeInf* and that M does not **SelectMa**-learn a branch of T one distinguishes the following two cases.

Case 1: There exists a stage s with $queue_s = \{\sigma_0, \dots, \sigma_q\}$ such that σ_0 is *never* removed from the queue in any stage $t > s$. Then the full tree ω^ω can be embedded into $\{\tau\colon \sigma\tau \in T\}$ and thus, T is in *TreeInf*. Futhermore, for all masters f on T, f extends σ_0, and $M(e, f[n]) = M(e, \sigma_0)$ for all $n \geq |\sigma_0|$. Hence, M converges on all masters to $j = M(e, \sigma_0)$. Then, φ_j is total. However, in this case, from some point on, we will only extend branches which are inconsistent with φ_j due to condition (c). Hence, φ_j is no branch of T. Contradiction.

Case 2: Every string is eventually removed from $queue_s$. In this case, the binary tree $\{0,1\}^\omega$ can be embedded into T, which again implies that T is in *TreeInf*. By construction, M will make infinitely many mind changes on every master of T due to line (7). Thus, for every master f of T, M fails to converge to a single program on input f. Contradiction. □

4 Master Learning versus Branch Learning

How is ordinary branch learning [12, 20], where the learner can only inspect the tree but has no access to a master, related to master learning? As already noted in the introduction, this comparison is, in contrast to the results of Sections 2 and 3, effected by the input model for the tree. In order to make the differences clear, we denote, for $\mathbf{Crit} \in \{\mathbf{ArbMaId}, \mathbf{ArbMa}, \mathbf{SelectMaId}, \mathbf{SelectMa}\}$, the version of **Crit**, where $M(e, f[n])$ in Definition 2.1 is replaced with $M(T[n], f[n])$, by **Enum-Crit**.

It is well known that, in general, enumerations of graphs are far less useful then programs. Therefore, it is not so surprising that $\mathbf{ArbMaId} \not\subseteq \mathbf{Enum\text{-}SelectMa}$. In other words, if one is working with an enumeration of the tree, then even the most powerful master learning notion fails to capture the most restrictive master learning notion working with programs of the trees. This result, in particular, implies that $\mathbf{Enum\text{-}Crit} \subset \mathbf{Crit}$ for all master learning criteria **Crit**.

The most powerful branch learning notion from [12] is called *weak* **Bc**-*learning*.[9] A class $\mathcal{C} \subseteq$ *Tree* is in **BranchWBc**, if there exists a Turing machine M such that for all $T \in \mathcal{C}$ and for almost all n, the function $\varphi_{M(T[n])}$ is an infinite computable branch of T, where the tree $T \subseteq \omega^*$ is identified with its characteristic function. Note that a **BranchWBc**-learner is allowed to make infinitely many *semantic* mind changes.

A machine *synthesizes* branches for a class $\mathcal{C} \subseteq$ *Tree in the limit* if it computes from every index of a tree $T \in \mathcal{C}$ a sequence of programs, which converges (syntactically) to a (program for an) infinite branch of T. The corresponding class is denoted by **SynthLim**.[10]

By using the trees from Section 1.2 one can show that no "pure branch learning class" captures any "master learning classes", demonstrating the extreme power of learning from masters:

[9] The criterion is not very restrictive (weak), but, then, many things can be learned with respect to it, so, from that perspective, it is powerful.

[10] **SynthLim** is equivalent to the learning criterion called **Uni**$[K]$ in [12, 20].

Theorem 4.1 **Enum-ArbMaId** $\not\subseteq$ (**BranchWBc** $\cup$ **SynthLim**).

Proof. The result follows by the class $\mathcal{C} = \{eT_e\colon\ e \in \omega\}$ where the trees T_e are defined as in Section 1.2. Basically the same argumentation as given there yields that $\mathcal{C} \notin$ **BranchWBc** and $\mathcal{C} \notin$ **SynthLim** (see also [20] for details).

So, it remains to show $\mathcal{C} \in$ **Enum-ArbMaId**. Recall that if f is an infinite branch of T_e then $f(1) = 0$ iff W_e is infinite. By construction, if W_e is finite then $k0^\omega$ is an infinite branch of T_e for every $k > |W_e|$, and if W_e is infinite then $f_e = 0a_0a_1\ldots$ with $a_m = \mu s[|W_{e,s}| > m]$ is a branch of T_e. Furthermore, if there is a branch on T_e extending e then this branch is unique for every k. Now, $\mathcal{C} \in$ **Enum-ArbMaId** is witnessed by the following procedure:

> *On input tree eT_e and master f on eT_e wait until $f[2]$ is read from the input master. Then $f(0) = e$. Let $k = f(1)$. If $k = 0$ output a program for ef_e. Otherwise, if $k > 0$, output a program for ek^ω.*

□

What can we say about the other direction, that is, which master learning classes capture which branch learning classes? First, one can easily show, that the master learning notions which are working with enumerations of trees are too restrictive:

Theorem 4.2
BranchWBc $\cap$ **SynthLim** $\not\subseteq$ **Enum-SelectMa**.

Proof. By **BranchBc** we denote the restriction of **BranchWBc**, where the learner has to converge semantically, that is $\mathcal{C} \subseteq$ *Tree* is in **BranchBc**, if there is a learning machine M such that for all $T \in C$

$$(\exists f \text{ on } T)(\exists n_0)(\forall n \geq n_0)[\varphi_{M(f[n])} = f].$$

For $f \in REC$ let $T_f = \{f[n]\colon\ n \in \omega\}$ be the tree which contains exactly the branch f. We choose a set $S \in \mathbf{Bc} - \mathbf{Ex}$ [7]. Then, obviously, the class $\mathcal{C}_S = \{T_f\colon\ f \in REC\}$ is in **BranchBc** which is a subclass of **BranchWBc** and **SynthLim** [12]. However, $\mathcal{C}_S \in$ **Enum-SelectMa** would imply $S \in \mathbf{Ex}$, which is a contradiction. □

However, by definition, **SynthLim** is a subset of **ArbMa**. One can show that this inclusion is proper. Hence, we have **SynthLim** $\subset$ **ArbMa** $\subset$ **SelectMaId** $\subset$ **SelectMa**. The classes **SynthLim** and **BranchWBc** are incomparable [12], that is, there are classes in **BranchWBc** for which one cannot synthesize branches in the limit. Thus, it is interesting to see whether one of the three master learning notions extending **SynthLim** are powerful enough to capture **BranchWBc**. It turns out that while **ArbMa** is still too restrictive to cover **BranchWBc**, the power of a **SelectMaId**-learner suffices. Thus, if a learner gets the index of the tree and can watch *the right* master, then the learner can improve its weak performance of infinitely many semantic mind changes to the very strong syntactic convergence to the master in the limit. The next and the last theorem of the paper formalizes this positive mind change complexity reduction result, which has an interesting simulation proof:

Theorem 4.3 **BranchWBc** $\subseteq$ **SelectMaId**.

Proof. Let $\mathcal{C}$ be in **BranchWBc**. Given an index of a tree $T \in \mathcal{C}$, the set E of all programs output by the learner is uniformly enumerable and depends only on the graph of T and not on the index of T. The indices in E define the following subtree $T' \subseteq T$:

$$\sigma \in T' \Leftrightarrow \sigma \in T \wedge (\exists e \in E)\, (\forall x \in dom(\sigma))\, [\varphi_e(x)\downarrow = \sigma(x)].$$

T' is enumerable but may not be computable. Furthermore, for all indices of T, the strings in T' are enumerated in the same order. In addition, T' has only finitely many finite branches, which cannot be extended to infinite branches, that is, almost all nodes of T' lie on an infinite branch of T'. So, knowing the enumeration of T', it is possible to extend almost every node σ of T' to an infinite branch of T'. This branch u_σ is defined inductively after initializing it with σ. Let now n be the first value not yet defined, then let

$$u_\sigma(n) = \begin{cases} a & \text{if } a \text{ is the first number such that } u_\sigma(0)u_\sigma(1)\dots u_\sigma(n-1)\,a \\ & \text{is enumerated into } T'; \\ \uparrow & \text{otherwise, if there is no such } a. \end{cases}$$

Every function u_σ is either an infinite branch of T' or is a finite branch which cannot be extended and differs from all infinite branches of T' at some value. So, the following two propositions hold:

$$\tau \prec \sigma \text{ and } u_\tau[n] \preceq u_\sigma \implies u_\tau(n)\downarrow \tag{8}$$

$$\sigma \preceq \tau \preceq u_\sigma \implies u_\sigma = u_\tau \tag{9}$$

Since the **BranchWBc**-learner produces at least one infinite branch, T' has at least one infinite branch. Moreover, if σ is a node of T' which is longer than every finite branch, then the function u_σ is infinite. Taking the master $f = u_\sigma$, the following algorithm learns this branch. The learner starts with u_λ. Whenever for the current τ an argument n is found with $u_\tau(n)\downarrow \neq f(n)$ then τ is extended to $f(0)f(1)\dots f(n)$ and a mind change to this new u_τ is made. After finitely many mind changes, $u_\tau = u_\sigma$ and the **SelectMaId**-learner succeeds. □

5 Future Work

As already mentioned in the introduction, the experts, which have been observed in the behavioral cloning experiments, are not playing necessarily winning strategies. Also master chess players do not win every play: instead of being classified into winners and losers, real players are ranked, more or less continuously, from bad to very good players. In the present basic study of watching masters we abstracted from this fact. But in future work it would be interesting to study learning from imperfect masters. At first instance, the crucial question for this research is, how to model imperfect masters. For example, one may consider masters which are playing finite variants of winning strategies. Or one may assume, that one of m input masters, or a majority of them, knows the best move in each situation. Moreover, for imperfect masters the problem of on-line learning is no longer trivial [12]! It would be interesting to investigate probabilistic learning from imperfect masters. The performance of an on-line learner can be measured by the number of lost plays, until it is eventually playing perfectly. Is there a connection between the quality of the input masters and the number of plays which an on-line learner loses?

The one-player immortality games given by a deterministic finite automaton, as described in Section 1.1, are just a special case of the well known two-player finite-state games [4, 15, 28]. In such games there always exist winning strategies which can be executed by a finite automaton. In [19] it is investigated whether one can *efficiently* learn strategies for one- and two-player *closed* finite-state games from membership and play queries, where membership queries involve asking whether a certain position is already a loss, and play queries involve asking whether a certain finite automaton implements a winning strategy. It would be interesting to apply the master learning concept to this situation. Can the time and query complexity of a learner be improved if the learner can ask queries about a winning automaton?

In Section 3 we have stated natural examples witnessing the separation of $\mathbf{ArbMa}^{m+1}$ and $\mathbf{ArbMa}^{m}$, and of $\mathbf{SelectMa}^{2}$ and $\mathbf{SelectMa}^{1}$. It would be interesting to look whether further natural examples separate some other levels of the $\mathbf{SelectMa}^{m}$ hierarchy.

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