Effectivizing Inseparability

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Abstract

Smullyan's notion of effectively inseparable pairs of sets is not the best effective/constructive analog of Kleene's notion of pairs of sets inseparable by a recursive set. We present a corrected notion of effectively inseparable pairs of sets, prove a characterization of our notion, and show that the pairs of *index sets* effectively inseparable in Smullyan's sense are the same as those effectively inseparable in ours. In fact we characterize the pairs of index sets effectively inseparable in either sense thereby generalizing Rice's Theorem. For *subrecursive* index sets we have sufficient conditions for various inseparabilities to hold. For inseparability by sets in the same subrecursive class we have a characterization. The latter essentially generalizes Kozen's (and Royer's later) Subrecursive Rice Theorem, and the proof of each result about subrecursive index sets is presented "Rogers style" with care to observe subrecursive restrictions.

There are pairs of sets effectively inseparable in Smullyan's sense, but not effectively inseparable in ours. The proof of this involves a non-effective construction by finite extensions with the unusual and interesting feature that alternate stages in the construction apply an instance of Smullyan's Double Recursion Theorem effective in the previous stage. Our construction yields as a corollary that the pairs of sets effectively inseparable in Smullyan's sense, but not in ours, are plentiful in the sense of Baire Category. By way of contrast with the previous result we show that, for pairs of *r.e. sets*, our notion and Smullyan's are coextensive. We call our notion *effective* Δ_1^0*-inseparability* and generalize it, Smullyan's notion, and all our results (except those about subrecursive index sets) to the Δ_n^0 level, for each $n > 1$. (Royer and Case apply effective Δ_2^0-inseparability to obtain results in structural complexity theory.) For subrecursive index sets we have a sufficient condition for effective Δ_2^0-inseparability. The proof of this latter result is made compact by an application of Royer and Case's Hybrid Recursion Theorem which facilitates interaction between a subrecursive programming system and a programming system for the partial limiting-recursive functions.

*This paper is dedicated to the memory of John Myhill. It grew out of a joint monograph with Royer, and owes an *especial* intellectual debt to that work and the methods in Myhill's paper on creative sets and Smullyan's monograph. A suggestion of Royer's provided a cleaner version of Theorem 3. He also made some helpful suggestions regarding presentation. The research was supported in part by NSF grant # CCR-8713846.

N denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Lower case letters, with or without decorations, near the front and rear of the alphabet range over N, and f, g, and h range over (total) functions with arguments and values from N. A, B, and C range over subsets of N, and $\overline{A}$ denotes the complement of A. $\mathrm{card}(\{A\})$ denotes the cardinality of A, and χ_A denotes the characteristic function of A, the function which is 1 on A and 0 on $\overline{A}$. Let φ denote a fixed acceptable programming system (numbering) for the partial recursive functions: $N \to N$ [Rog58, Blu67, MY78, Ric80, Ric81, Roy87], and let W_p denote the domain of φ_p. W_p is, then, the r.e. set ($\subseteq N$) accepted by φ-program p. Following Meyer we let $\downarrow$ denote convergence of a computation, and $\uparrow$ denote divergence [Rog87]. Let $\lambda x, y\,.\,\langle x, y\rangle$ denote a fixed pairing function (a recursive, bijective mapping: $N \times N \to N$ [Rog87]) with respective inverses π_1 and π_2. $\mathcal{L}$ineartime denotes the class of functions computable on multi-tape Turing machines [HU79] within a linear time bound of the length of the input (in binary). We suppose that $\lambda x, y\,.\,\langle x, y\rangle$, π_1, and π_2 are each in $\mathcal{L}$ineartime. An example based on bit interlacing with standard binary representations [BL74] of numbers is provided in [RC86, RC89]. Let $\lambda x_1, \ldots, x_n\,.\,\langle x_1, \ldots, x_n\rangle$ be a fixed recursive bijection: $N^n \to N$ based on $\lambda x, y\,.\,\langle x, y\rangle$. For example, we could take $\langle x_1, x_2, x_3\rangle = \langle x_1, \langle x_2, x_3\rangle\rangle$. We let Φ denote a fixed Blum complexity measure associated with φ [Blu67, MY78, DW83]. $\Phi_p(z)$ is, intuitively, the run time of φ-program p on input z. We say that a number w *appears in* W_x *in exactly* z (respectively, $\leq z$) *steps* $\Leftrightarrow$ $\Phi_x(w) = z$ (respectively, $\leq z$). A number w *appears in* W_x *before* W_y $\Leftrightarrow$ $\Phi_x(w)\!\downarrow\, < \Phi_y(w) \leq \infty$. A number w *appears in* W_x *at the same time as it appears in* W_y $\Leftrightarrow$ $\Phi_x(w) = \Phi_y(w)\!\downarrow\, < \infty$. $f|_A$ denotes the graph of f restricted to the set A. $f|_{<x}$ denotes $f|_{\{w|w<x\}}$, and $f|_{\geq x}$ denotes $f|_{\{w|w\geq x\}}$. $A \triangle B$ denotes $(A - B) \cup (B - A)$, the symmetric difference of A and B. Any other unexplained notation or terminology is from [Rog87]. For example, Δ^0_n, Σ^0_n, and Π^0_n represent the usual levels in the *arithmetical* hierarchy.

We provide a brief, partial history of the notions that preceded the subject of this paper. Gödel's First Incompleteness Theorem [Göd86, Men79] indirectly motivated Dekker's notion of *productive set* [Dek55, Rog87]. This notion was based on Post's earlier notion of *creative set* [Pos44, Myh55, Rog87]. Gödel's Theorem directly motivated Kleene's notion of *recursively inseparable sets* [Kle52, Rog87]. We explain these notions just below.

A is *productive* $\Leftrightarrow$ there is an effective procedure which, given any x such that $W_x \subseteq A$, returns a value in $(A - W_x)$. Here is an example from [Rog87] extracting a recursion-theoretic essence of Gödel's Theorem. Gödel number the set of sentences of arithmetic onto N and identify sentences with their Gödel numbers. The set of *true* sentences of arithmetic is productive. Post [Pos44] was interested in productive sets with r.e. complements, for example, the set of sentences not provable in Peano Arithmetic.

A is *recursively inseparable from* B $\Leftrightarrow$ A is disjoint from B and there is no recursive set C such that $A \subseteq C \subseteq \overline{B}$. (The reader may find a simple Euler-Venn diagram helpful to see the relation between the sets in this definition and others of this paper.) Kleene [Kle52, Rog87] noted that the set of sentences P provable in Peano Arithmetic is recursively inseparable from the set of sentences R refutable in Peano Arithmetic. If a complete, *recursive* axiomatization of arithmetic existed, its deductive closure C would be a recursive set separating P from R.

Dekker [Dek55, Rog87] essentially defined a set A to be *completely productive* (abbreviated: *c-productive*) $\Leftrightarrow$ $(\exists$ recursive $f)(\forall x)[f(x) \in W_x \triangle A]$. Myhill [Dek55, Rog87] showed that the c-productive and the productive sets coincide. Intuitively, A is c-productive $\Leftrightarrow$ there is an effective procedure to find, given any x, a counter-example to '$W_x = A$'. The c-productive sets are, then, those sets that fail to be r.e. in a certain effective sense.

Now, to get to a central concept of this paper, just as there is an effective sense in which sets can fail to be r.e., there are effective senses of one set failing to be recursively separable from another. For example, we define just below *effective Δ^0_1-inseparability.* Intuitively, A is *effectively Δ^0_1-inseparable from B* $\Leftrightarrow$ [A is disjoint from B and there is an effective procedure to find, given x and y, a counter-example to '$[A \subseteq W_x = \overline{W_y} \subseteq \overline{B}]$'].

Definition 1 *A is* effectively Δ^0_1-inseparable from B $\Leftrightarrow$ $[(A \cap B) = \emptyset \wedge (\exists$ *recursive* $f)(\forall x, y)[$ $f(x, y) \in ((W_x \cap B) \cup (\overline{W_x} \cap A) \cup (W_y \cap A) \cup (\overline{W_y} \cap B) \cup (W_x \triangle \overline{W_y}))]]$.

Here is a characterization of effective Δ^0_1-inseparability.

Theorem 1 *A is effectively Δ^0_1-inseparable from B* $\Leftrightarrow$ $[(A \cap B) = \emptyset \wedge (\exists$ *recursive* $f)(\forall x, y)[$ $f(x, y) \in ((W_x \cap B) \cup (W_y \cap A) \cup (\overline{W_x} \cap \overline{W_y}))]]$.

PROOF. ($\Leftarrow$) is immediate.

($\Rightarrow$): Suppose A is effectively Δ^0_1-inseparable from B as witnessed by the recursive function f.

By Smullyan's Double Parametric Recursion Theorem [Smu61, Rog87], there are recursive functions g and h such that, for all x and y,

$$W_{g(x,y)} = \begin{cases} \{f(g(x,y), h(x,y))\}, & \text{if } f(g(x,y), h(x,y)) \text{ appears in } W_x \text{ before } W_y; \\ \emptyset, & \text{otherwise;} \end{cases}$$

$$W_{h(x,y)} = \begin{cases} \{f(g(x,y), h(x,y))\}, & \text{if } f(g(x,y), h(x,y)) \text{ appears in } W_y \text{ before (or} \\ & \quad \text{at the same time as it appears in) } W_x; \\ \emptyset, & \text{otherwise.} \end{cases}$$

We show that $(\forall x, y)[f(g(x, y), h(x, y)) \in ((W_x \cap B) \cup (W_y \cap A) \cup (\overline{W_x} \cap \overline{W_y}))]$.

Case (1). $f(g(x, y), h(x, y))$ appears in W_x before W_y. Then

$$W_{g(x,y)} = \{f(g(x, y), h(x, y))\} \text{ and } W_{h(x,y)} = \emptyset.$$

Hence, $f(g(x, y), h(x, y)) \notin ((\overline{W_{g(x,y)}} \cap A) \cup (A \cap W_{h(x,y)}) \cup (W_{g(x,y)} \triangle \overline{W_{h(x,y)}}))$. Therefore, since f witnesses the effective Δ^0_1-inseparability of A from B, $f(g(x, y), h(x, y)) \in B$. Hence, $f(g(x, y), h(x, y)) \in (W_x \cap B)$.

Case (2). $f(g(x, y), h(x, y))$ appears in W_y before (or at the same time as it appears in) W_x. Then, by an argument symmetric to that of Case (1), $f(g(x, y), h(x, y)) \in (W_y \cap A)$.

Case (3). Neither Cases (1) nor (2). Then, $f(g(x, y), h(x, y)) \in (\overline{W_x} \cap \overline{W_y})$. $\square$

Smullyan [Smu61] effectivized Kleene's notion of recursive inseparability in an intensionally, and, as we shall see (Theorem 6), extensionally, different way. Definition 2 is essentially Smullyan's definition.

Definition 2 (Smullyan) *A is* effectively inseparable from $B \Leftrightarrow [(A \cap B) = \emptyset \wedge (\exists$ *recursive* $f)(\forall x, y)[A \subseteq W_x \subseteq \overline{W_y} \subseteq \overline{B} \Rightarrow f(x, y) \in (\overline{W_x} \cap \overline{W_y})]]$.

If $\mathbf{C}$ is a class of partial recursive functions, then $\mathcal{P}_{\mathbf{C}}$ denotes $\{p \mid \varphi_p \in \mathbf{C}\}$, the index set [Rog87] determined by $\mathbf{C}$. The next theorem characterizes effectively Δ_1^0-inseparable pairs of *index sets* and implies that for pairs of index sets our notion and Smullyan's agree. (Results related to special cases may be found in [DM58, Hay65].)

Theorem 2 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are disjoint classes of partial recursive functions. Then (a) through (d) are equivalent.*

(a) $\mathcal{P}_{\mathbf{C}}$ *is effectively* Δ_1^0*-inseparable from* $\mathcal{P}_{\mathbf{D}}$*.*

(b) $\mathcal{P}_{\mathbf{C}}$ *is effectively inseparable from* $\mathcal{P}_{\mathbf{D}}$*.*

(c) $\mathcal{P}_{\mathbf{C}}$ *is recursively inseparable from* $\mathcal{P}_{\mathbf{D}}$*.*

(d) Both $\mathcal{P}_{\mathbf{C}}$ *and* $\mathcal{P}_{\mathbf{D}}$ *are non-empty.*

PROOF. Suppose the hypothesis. If either of $\mathcal{P}_{\mathbf{C}}$ or $\mathcal{P}_{\mathbf{D}}$ is empty, clearly they are separated by a recursive set.

Suppose, then, $\varphi_c \in \mathbf{C}$ and $\varphi_d \in \mathbf{D}$. By Definitions 1 and 2, it suffices to show $(\exists$ recursive $f)(\forall x, y)[f(x, y) \in ((W_x \cap \mathcal{P}_{\mathbf{D}}) \cup (W_y \cap \mathcal{P}_{\mathbf{C}}) \cup (\overline{W_x} \cap \overline{W_y}))]$.

By Kleene's Parametric Recursion Theorem [Rog87, Ric80, Ric81, Roy87], there is a recursive function f such that, for all x, y, and z,

$$\varphi_{f(x,y)}(z) = \begin{cases} \varphi_d(z), & \text{if } f(x,y) \text{ appears in } W_x \text{ before } W_y; \\ \varphi_c(z), & \text{if } f(x,y) \text{ appears in } W_y \text{ before (or at} \\ & \quad \text{the same time as it appears in) } W_x; \\ \uparrow, & \text{otherwise.} \end{cases}$$

Case (1). $f(x, y)$ appears in W_x before W_y. Then, $\varphi_{f(x,y)} = \varphi_d$; hence, $f(x, y) \in \mathcal{P}_{\mathbf{D}}$. Therefore, $f(x, y) \in (W_x \cap \mathcal{P}_{\mathbf{D}})$.

Case (2). $f(x, y)$ appears in W_y before (or at the same time as it appears in) W_x. Then, by an argument symmetric to that of Case (1), $f(x, y) \in (W_y \cap \mathcal{P}_{\mathbf{C}})$.

Case (3). Neither Cases (1) nor (2). Then, $f(x, y) \in (\overline{W_x} \cap \overline{W_y})$. □

Theorem 2 obviously generalizes Rice's Theorem [Ric53, Rog87, DW83].

Theorem 3 below provides a sufficient condition for disjoint pairs of *subrecursive index sets* (defined two paragraphs below) to be effectively Δ_1^0-inseparable. We provide next the preliminaries for that theorem and other results about subrecursive index sets.

A *subrecursive class* is an r.e. class of (total) recursive functions: $N \to N$. Suppose $\mathbf{S}$ is a subrecursive class. ψ is a *programming system* (or *effective numbering*) *for* $\mathbf{S} \Leftrightarrow \psi$ is a recursive function such that $\mathbf{S} = \{\lambda z \centerdot \psi(r, z) \mid r \in N\}$. If ψ is a programming system for a subrecursive class, we write ψ_r for $\lambda z \centerdot \psi(r, z)$ and speak of r as a *ψ-program* for ψ_r. Generally subrecursive classes studied in computer science are explicitly based on bounding the time or space complexity of the functions allowed in the class; those studied in mathematics are implicitly so based [CB71, Con71, Ros84, RC86, RC89]. For example, $\mathcal{P}$olytime, the class of functions each computable on a multi-tape Turing machine within time given by a polynomial in the length of the input (in binary), is a subrecursive class. If one Gödel numbers (onto N) the multi-tape Turing machines each explicitly clocked to halt within time bounded by some polynomial in the length of the input and sets $\psi_r =$ the function computed by the machine with Gödel number r, then ψ is a subrecursive programming system for $\mathcal{P}$olytime [BH79, HB79, Koz80, RC86, RC89].

Suppose $\mathbf{S}$ and ψ are fixed such that $\mathbf{S}$ is a subrecursive class and ψ is a subrecursive programming system for $\mathbf{S}$. If $\mathbf{C} \subseteq \mathbf{S}$, then $\mathcal{S}_{\mathbf{C}}$ denotes $\{r \mid \psi_r \in \mathbf{C}\}$, the *subrecursive index set determined by* $\mathbf{C}$ [Koz80].

We make two basic assumptions about $\mathbf{S}$ and ψ.

Assumption 1 $\mathbf{S}$ *contains $\mathcal{L}$ineartime and it is closed under (inner and outer) composition with functions in $\mathcal{L}$ineartime.*

We present some consequences of Assumption 1.

Recall from above that we chose $\lambda x, y \centerdot \langle x, y\rangle$, π_1, and π_2 in $\mathcal{L}$ineartime. Thanks to Assumption 1, these are clearly useful to simulate the effect of having functions of multiple arguments in $\mathbf{S}$. As another example, let

$$\text{cond} = \lambda\langle x, y, z\rangle \centerdot \begin{cases} y, & \text{if } x > 0; \\ z, & \text{if } x = 0. \end{cases}$$

Then cond is clearly in $\mathcal{L}$ineartime. Hence, $\mathbf{S}$ is closed under definition by cases. This is exploited below in the proof of Theorem 3 (and also the proofs of Theorems 4, 5, and 8 and Remarks 1 and 3). Let $\max = \lambda\langle x_1, \ldots, x_n\rangle \centerdot$[the maximum of $x_1, \ldots, x_n$]. Clearly max is in $\mathcal{L}$ineartime. Hence, $\mathbf{S}$ is also closed under taking the maximum of a constant number of functions. We exploit this in the proof of Theorem 4 below. We also note (and exploit below) that, in effect, the Boolean functions $\wedge$ (and), $\vee$ (or), and $\neg$ (not) are in $\mathcal{L}$ineartime.

Let $\mathcal{Q}$uadratictime be the class of functions: $N \to N$ computable within time bounded by some degree-two polynomial in the length of the input. Then $\mathbf{S} = \mathcal{Q}$uadratictime satisfies Assumption 1, but is *not* itself closed under composition. The same is true for $\mathbf{S} = \mathcal{C}$ubictime, defined in the obvious way, etc.

Assumption 2 ψ *satisfies the Kleene s-m-n Theorem with an s-1-1 function in $\mathcal{L}$ineartime [Koz80, Roy87, RC86, RC89].*

Essentially s-1-1 provides *substitution* of data into programs, of course filtered through the Gödel numbering. This is an *extremely* elementary operation if an *efficient* Gödel numbering is chosen. Therefore, it is quite plausible to be able to perform this operation in linear time for a "natural", efficiently numbered, subrecursive programming system for a class $\mathbf{S}$ satisfying our assumptions. Marcoux [Mar89] shows that, as a control structure [Ric80, Ric81, Ric82, Roy87], s-1-1 is more fundamental than *composition* with respect to instance complexity. [RC86, RC89] (generalizing [Koz80]) provide extremely simple sufficient conditions for ψ to have an instance of an s-1-1 function in $\mathcal{L}$ineartime. Now, the Kleene form of his

Strong Recursion Theorem [Rog87, Page 214], but *not* the Rogers Pseudo-Fixed Point form [Rog87, Page 180] (see also [Ric80, Ric81, Ric82, Roy87]), holds in most naturally occurring *subrecursive* programming systems computing at least the functions in $\mathcal{L}$ineartime [Koz80, Roy87, RC86, RC89]. In fact, it easily follows from our assumptions on $\mathbf{S}$ and ψ that ψ has a $\mathcal{L}$ineartime-effective Parametric Recursion Theorem [RC86, RC89] ('$\mathcal{L}$ineartime-effective' merely means that the witnessing function is in $\mathcal{L}$ineartime). Our proof below of Theorem 3 contains an application of this recursion theorem. Royer and Case [RC86, RC89] show how to efficiently Gödel number multi-tape Turing machines and formulate clocking mechanisms so that explicitly clocked systems for $\mathcal{L}$ineartime, $\mathcal{Q}$uadratictime, ..., $\mathcal{P}$olytime, etc., based on this numbering of Turing machines, satisfy the assumptions above. They also present an axiomatic treatment of clocked systems and show how these inherit, from the underlying general purpose machines, relatively efficient control structures [Roy87] involving s-1-1, composition, recursion theorems, etc.

Theorem 3 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are disjoint subsets of* $\mathbf{S}$. *Suppose there are* ψ*-programs* b, c, *and* d *such that, for every* t,

$$\psi_b|_{<t} \subset \lambda z \centerdot \psi_c(\langle t, z\rangle) \in \mathbf{C} \textit{ and } \psi_b|_{<t} \subset \lambda z \centerdot \psi_d(\langle t, z\rangle) \in \mathbf{D}.$$

Then: $\mathcal{S}_{\mathbf{C}}$ *is effectively* Δ^0_1*-inseparable from* $\mathcal{S}_{\mathbf{D}}$.

We noted above that $\mathbf{S} = \mathcal{C}$ubictime satisfies our Assumption 1. If we suppose ψ is a subrecursive programming system for this class satisfying Assumption 2, then, by Theorem 3, $\{r \mid \psi_r \in (\mathcal{Q}\text{uadratictime} - \mathcal{L}\text{ineartime})\}$ is effectively Δ^0_1-inseparable from $\{r \mid \psi_r \in \mathcal{L}\text{ineartime}\}$; $\{r \mid (\exists t)[\psi_r = (\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 1|_{\geq t})]\}$ is effectively Δ^0_1-inseparable from $\{r \mid \psi_r = \lambda z \centerdot 0\}$; and $\{r \mid (\exists t)[\psi_r = (\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 1|_{\geq t})]\}$ is effectively Δ^0_1-inseparable from $\{r \mid (\exists t)[\psi_r = (\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 2|_{\geq t})]\}$.

PROOF OF THEOREM 3. Suppose the hypotheses.

For this proof we take Φ to be a special, "delayed" Blum complexity measure such that one can compute in $\mathcal{L}$ineartime both $\lambda p, z, t \centerdot [\Phi_p(z) \leq t]$ and

$$\lambda p, z, t \centerdot \begin{cases} \Phi_p(z), & \text{if } \Phi_p(z) \leq t; \\ t+1, & \text{otherwise.} \end{cases}$$

Royer and Case [RC86, RC89] show that, for any φ, such an associated Φ always exists. In the rhetoric below in this proof (and others which employ such a special Φ), locutions of the form, "so-and-so appears in W_p with such-and-such restrictions on the number of steps," implicitly refer, then, to this special Φ.

By the $\mathcal{L}$ineartime-effective Kleene Parametric Recursion Theorem for ψ [RC86, RC89], there is a $\mathcal{L}$ineartime function f such that, for all x, y, and z,

$$\psi_{f(x,y)}(z) = \begin{cases} \psi_d(\langle t, z\rangle), & \text{if } f(x,y) \text{ appears in } W_x \text{ in exactly } t \text{ steps, where} \\ & \quad t \leq z, \text{ and } f(x,y) \text{ appears in } W_x \text{ before } W_y; \\ \psi_c(\langle t, z\rangle), & \text{if } f(x,y) \text{ appears in } W_y \text{ in exactly } t \text{ steps, where} \\ & \quad t \leq z, \text{ and } f(x,y) \text{ appears in } W_y \text{ before (or at;} \\ & \quad \text{the same time as it appears in) } W_x; \\ \psi_b(z), & \text{otherwise.} \end{cases}$$

The rest of the proof is a straightforward modification of that of Theorem 2. $\square$

In Theorem 4 just below we make the sufficient condition of Theorem 3 somewhat less constructive to obtain a sufficient condition for *recursive inseparability* of subrecursive index sets.

Theorem 4 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are disjoint subsets of* $\mathbf{S}$*. Suppose, for each* i*,* j*, and* k *with* $i \leq \ell$*,* $j \leq m$*, and* $k \leq n$*, there are corresponding* ψ*-programs* b_i*,* c_j*, and* d_k *such that, for every* t*, there are corresponding* i*,* j*, and* k*, with* $i \leq \ell$*,* $j \leq m$*, and* $k \leq n$*, for which*

$$\psi_{b_i}|_{<t} \subset \lambda z \boldsymbol{.} \psi_{c_j}(\langle t, z\rangle) \in \mathbf{C} \text{ and } \psi_{b_i}|_{<t} \subset \lambda z \boldsymbol{.} \psi_{d_k}(\langle t, z\rangle) \in \mathbf{D}.$$

Then: $\mathcal{S}_{\mathbf{C}}$ *is recursively inseparable from* $\mathcal{S}_{\mathbf{D}}$*.*

Remark 1 *For both Theorems 3 and 4, the sufficient conditions are not necessary.*

Remark 2 *The sufficient condition of Theorem 4 does* not *imply that* $\mathcal{S}_{\mathbf{C}}$ *is effectively* Δ^0_1*-inseparable from* $\mathcal{S}_{\mathbf{D}}$*.*

We prove Theorem 4, Remark 1, and Remark 2, in that order.

Proof of Theorem 4. Suppose the hypotheses. For this proof only we make the convention that i, j, and k (with or without decorations) are restricted in range thus: $i \leq \ell$, $j \leq m$, and $k \leq n$.

Also for this proof we again take Φ to be a special, "delayed" Blum complexity measure such that one can compute in $\mathcal{L}$ineartime both $\lambda p, z, t \boldsymbol{.} [\Phi_p(z) \leq t]$ and

$$\lambda p, z, t \boldsymbol{.} \begin{cases} \Phi_p(z), & \text{if } \Phi_p(z) \leq t; \\ t+1, & \text{otherwise.} \end{cases}$$

As we noted in the proof of Theorem 3, Royer and Case [RC86, RC89] show that, for any φ, such an associated Φ always exists.

Let x and y be fixed. By the $((\ell+1)\cdot(m+1)\cdot(n+1))$-ary Recursion Theorem for ψ [RC86, RC89], there are $((\ell+1)\cdot(m+1)\cdot(n+1))$ self-other referential ψ-programs $e_{i,j,k}$ such that, for all z,

$$\psi_{e_{i,j,k}}(z) = \begin{cases} \psi_{d_k}(\langle t, z\rangle), & \text{if } \textit{each } e_{i',j',k'} \text{ appears in } \textit{either } W_x \textit{ or } W_y \text{ within a} \\ & \quad \text{number of steps least upper bounded by } t, \text{ where} \\ & \quad t \leq z, \text{ and } e_{i,j,k} \text{ itself appears in } W_x \text{ before } W_y; \\ \psi_{c_j}(\langle t, z\rangle), & \text{if } \textit{each } e_{i',j',k'} \text{ appears in } \textit{either } W_x \textit{ or } W_y \text{ within a} \\ & \quad \text{number of steps least upper bounded by } t, \text{ where} \\ & \quad t \leq z, \text{ and } e_{i,j,k} \text{ itself appears in } W_y \text{ before (or at} \\ & \quad \text{the same time as it appears in) } W_x; \\ \psi_{b_i}(z), & \text{otherwise.} \end{cases} \tag{1}$$

It suffices to show that, for *some* triple (i,j,k),

$$e_{i,j,k} \in ((W_x \cap \mathcal{S}_{\mathbf{D}}) \cup (W_y \cap \mathcal{S}_{\mathbf{C}}) \cup (\overline{W_x} \cap \overline{W_y})).$$

Case (1). Each $e_{i',j',k'}$ appears in either W_x or W_y. Let t be the least upper bound of the number of steps required for each $e_{i',j',k'}$ to so appear. By the hypotheses of the theorem, there is, associated with this t a triple (i,j,k) such that

$$\psi_{b_i}|_{<t} \subset \lambda z \centerdot \psi_{c_j}(\langle t, z\rangle) \in \mathbf{C} \ \text{ and } \ \psi_{b_i}|_{<t} \subset \lambda z \centerdot \psi_{d_k}(\langle t, z\rangle) \in \mathbf{D}. \tag{2}$$

Consider, then, $e_{i,j,k}$.

Subcase (1.1). $e_{i,j,k}$ appears in W_x before W_y. Then by (1),

$$\psi_{e_{i,j,k}} = (\psi_{b_i}|_{<t} \cup \lambda z \centerdot \psi_{d_k}(\langle t, z\rangle)|_{\geq t});$$

hence, by (2),

$$\psi_{e_{i,j,k}} = \lambda z \centerdot \psi_{d_k}(\langle t, z\rangle) \in \mathcal{S}_{\mathbf{D}}.$$

Therefore, $e_{i,j,k} \in (W_x \cap \mathcal{S}_{\mathbf{D}})$.

Subcase (1.2). $e_{i,j,k}$ appears in W_y before (or at the same time as it appears in) W_x. Then by an argument symmetric to that of Subcase (1.1), $e_{i,j,k} \in (W_y \cap \mathcal{S}_{\mathbf{C}})$.

Case (2). Not Case (1). Then, for some triple (i,j,k), $e_{i,j,k} \in (\overline{W_x} \cap \overline{W_y})$. □

Proof of Remark 1. We take Φ to be a special, "delayed" Blum complexity measure as in the proofs of Theorems 3 and 4. Let $\mathbf{C} = \{\psi_r \mid (\exists t)[\psi_{\psi_r(0)} = (\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 1|_{\geq t})]\}$ and $\mathbf{D} = \{\psi_r \mid (\exists t)[\psi_{\psi_r(0)} = (\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 2|_{\geq t})]\}$. $\mathbf{C}$ and $\mathbf{D}$ are non-trivial since $\mathbf{S}$ contains $\mathcal{L}$ineartime, and, clearly, $\mathbf{C}$ and $\mathbf{D}$ are disjoint subsets of $\mathbf{S}$ that do not satisfy the hypotheses of Theorems 3 and 4. We show that, nonetheless, $\mathcal{S}_{\mathbf{C}}$ is effectively Δ^0_1-inseparable from $\mathcal{S}_{\mathbf{D}}$. The techniques of [RC86, RC89] easily establish a $\mathcal{L}$ineartime-effective Delayed Recursion Theorem [Cas74] for ψ. Hence, there is a $\mathcal{L}$ineartime function f such that, for all w, x, y, and z,

$$\psi_{\psi_{f(x,y)}(w)}(z) = \begin{cases} 2, & \text{if } f(x,y) \text{ appears in } W_x \text{ in } \leq z \text{ steps,} \\ & \quad \text{and } f(x,y) \text{ appears in } W_x \text{ before } W_y; \\ 1, & \text{if } f(x,y) \text{ appears in } W_y \text{ in } \leq z \text{ steps,} \\ & \quad \text{and } f(x,y) \text{ appears in } W_y \text{ before (or at;} \\ & \quad \text{the same time as it appears in) } W_x; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to argue, much as in the proofs of previous theorems, that, for all x and y,

$$f(x,y) \in ((W_x \cap \mathcal{S}_{\mathbf{D}}) \cup (W_y \cap \mathcal{S}_{\mathbf{C}}) \cup (\overline{W_x} \cap \overline{W_y})).$$

□

Proof of Remark 2. We adapt a trick of Fulk's from [CFE83]. Let A be a set such that both A and $\overline{A}$ are immune [Rog87, Page 108]. Let $\mathbf{C} = \{\lambda z \centerdot 0\}$. Let

$$\mathbf{D} = \{(\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 1|_{\geq t}) \mid t \in A\} \cup \{(\lambda z \centerdot 0|_{<t} \cup \lambda z \centerdot 2|_{\geq t}) \mid t \in \overline{A}\}.$$

Since $\mathcal{L}$ineartime $\subseteq \mathbf{S}$, we can choose ψ-program(s) b_0 and c_0 such that $\psi_{b_0} = \psi_{c_0} = \lambda z \centerdot 0$. Furthermore, for each $k \in \{0, 1\}$, we can choose a ψ-program d_k such that

$$\psi_{d_k} = \lambda \langle t, z \rangle \centerdot \begin{cases} 0, & \text{if } z < t; \\ k+1, & \text{otherwise.} \end{cases}$$

Clearly, then, $\mathbf{C}$, $\mathbf{D}$, b_0, c_0, d_0, and d_1 satisfy the hypotheses of Theorem 4. Suppose for contradiction that f is a recursive function such that

$$(\forall x, y)[f(x, y) \in ((W_x \cap \mathcal{S}_{\mathbf{D}}) \cup (W_y \cap \mathcal{S}_{\mathbf{C}}) \cup (\overline{W_x} \cap \overline{W_y}))]. \tag{3}$$

By the *s-m-n* Theorem for φ, there is a recursive function g such that, for all s,

$$W_{g(s)} = \{r \mid \lambda y \centerdot 0|_{<s} \subset \psi_r\}.$$

Let q be a φ-program such that

$$W_q = \{r \mid (\exists z)[\psi_r(z) > 0\}.$$

Clearly, $(W_q \cap \mathcal{S}_{\mathbf{C}}) = \emptyset$ and $(\forall s)[(\overline{W_{g(s)}} \cap \overline{W_q}) = \emptyset]$. Hence, by (3),

$$(\forall s)[f(g(s), q) \in (W_{g(s)} \cap \mathcal{S}_{\mathbf{D}})]. \tag{4}$$

Now, by (4), for each s, $s \leq \text{card}(\{z \mid \psi_{f(g(s),q)}(z) = 0\}) < \infty$. Therefore, $\{\psi_{f(g(s),q)} \mid s \in N\}$ is infinite. Let $\tau(r) = \min_t[(\exists v > 0)[\psi_r(t) = v]$. τ is clearly partial recursive. For each $v \in \{1, 2\}$, let $A_v = \{\tau(f(g(s), q)) \mid v \in \text{range}(\psi_{f(g(s),q)})\}$. Clearly, then, either A_1 is an infinite r.e. subset of A or A_2 is an infinite r.e. subset of $\overline{A}$. This is a contradiction. Hence, by Theorem 1, $\mathcal{S}_{\mathbf{C}}$ is not effectively Δ^0_1-inseparable from $\mathcal{S}_{\mathbf{D}}$. □

We next proceed to define inseparability by *subrecursive* sets, where the class of *subrecursive sets associated with a subrecursive class* $\mathbf{S}$ (called the class of $\mathbf{S}$-*sets*) is

$$\{C \mid (\exists g \in \mathbf{S})[C = g^{-1}(1)]\}.$$

A special case occurs when $g \in \mathbf{S}$ is χ_C for some C. Then $g^{-1}(1) = C$, and, so, C is an $\mathbf{S}$-set.

Definition 3 *A is* $\mathbf{S}$-inseparable *from B* $\Leftrightarrow$ *[A and B are disjoint and there is no $\mathbf{S}$-set C such that $A \subseteq C \subseteq \overline{B}$].*

Theorem 5 just below characterizes $\mathbf{S}$-inseparability of associated subrecursive index sets. It also holds with the even less restrictive assumptions on $\mathbf{S}$ and ψ Royer [Roy87, Page 173] employed for his Subrecursive Rice Theorem (together with the assumption that $\chi_\emptyset$ and $\chi_N \in \mathbf{S}$).

Theorem 5 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are disjoint subsets of* $\mathbf{S}$. *Then (a) and (b) are equivalent.*
(a) $\mathcal{S}_{\mathbf{C}}$ *is* $\mathbf{S}$*-inseparable from* $\mathcal{S}_{\mathbf{D}}$.
(b) Both $\mathcal{S}_{\mathbf{C}}$ *and* $\mathcal{S}_{\mathbf{D}}$ *are non-empty.*

PROOF. Suppose the hypotheses. We note that $\emptyset$ and N are $\mathbf{S}$-sets since their characteristic functions are computable in linear time. If either of $\mathcal{S}_{\mathbf{C}}$ or $\mathcal{S}_{\mathbf{D}}$ is empty, clearly they are separated by an $\mathbf{S}$-set, namely, either $\emptyset$ or N.

Suppose, then, $\psi_c \in \mathbf{C}$ and $\psi_d \in \mathbf{D}$. Suppose C is an $\mathbf{S}$-set.

By the Kleene Recursion Theorem for ψ [RC86, RC89], there is a ψ-program e such that, for all x, y, and z,

$$\psi_e(z) = \begin{cases} \psi_d(z), & \text{if } e \in C; \\ \psi_c(z), & \text{if } e \notin C. \end{cases}$$

Suppose for contradiction that $\mathcal{S}_{\mathbf{C}} \subseteq C \subseteq \overline{\mathcal{S}_{\mathbf{D}}}$. Hence, $e \in C \Rightarrow \psi_e = \psi_d \in \mathbf{D}, \Rightarrow e \in \mathcal{S}_{\mathbf{D}}, \Rightarrow e \notin C, \Rightarrow \psi_e = \psi_c \in \mathbf{C}, \Rightarrow e \in \mathcal{S}_{\mathbf{C}}, \Rightarrow e \in C$. Therefore, $e \in C \Leftrightarrow e \notin C$, a contradiction. $\square$

Theorem 5 essentially generalizes Kozen's [Koz80] (and Royer's later [Roy87, Pages 173-174]) Subrecursive Rice Theorem.

Case introduced the notions of *r.e. inseparability* and *effective r.e. inseparability* [CFE83]. Some of the theorems of [CFE83] are lifted to Scott's CPO's [Sco70] in [Spr83]. A is said to be *r.e. inseparable from* $B \Leftrightarrow$ [A and B are disjoint and there is no r.e. set C such that $A \subseteq C \subseteq \overline{B}$]. A is *effectively r.e. inseparable from* $B \Leftrightarrow [A \cap B = \emptyset \wedge (\exists \text{ recursive } f)(\forall x)[f(x) \in ((A \cap \overline{W_x}) \cup (W_x \cap B))]]$. Royer [Roy89] applies both these notions to characterize the presence of proof speed-up between theories. Royer and Case [RC89] apply the lift to the Σ_2^0 level (from the r.e. $= \Sigma_1^0$ level) of both these notions to characterize relative program succinctness phenomena between subrecursive/complexity-bounded programming systems [Koz80, Roy87] and to obtain a tight incompleteness theorem about subrecursive program succinctness coupled with information loss.

Theorem 6 *There are disjoint sets* A *and* B *such that* A *is r.e. inseparable from* B *(therefore,* A *is* vacuously *effectively inseparable from* B*), but* A *is* not *effectively* Δ_1^0*-inseparable from* B.

PROOF. A and B are obtained by a *non*-effective construction by finite extensions in consecutive stages $s \geq 0$. A^s and B^s denote, respectively, the finitely much of A and B, respectively, determined by the beginning of *stage* s. A^0 and B^0 are both empty. A and B are constructed to explicitly satisfy, for each s, the requirements $\mathcal{R}_s^0$ and $\mathcal{R}_s^1$ below, where $\mathcal{R}_s^0$ is satisfied during *stage* $2s$ and $\mathcal{R}_s^1$ during *stage* $2s+1$. The finding of x_s and y_s in *stage* $2s+1$ is effective in s and (*canonical indices* [Rog87] for) A^{2s+1} and B^{2s+1} by Smullyan's Double Recursion Theorem. The construction ensures that, if a number is put into one of A and B, it is excluded from the other; hence, $(A \cap B) = \emptyset$.

$\mathcal{R}_s^0$: $(\exists w_s)[w_s \in ((W_s \cap B) \cup (\overline{W_s} \cap A))]$.

$\mathcal{R}_s^1$: $[\varphi_s\, total \Rightarrow (\exists x_s, y_s)[$
$[\varphi_s(\langle x_s, y_s\rangle) \in (W_{x_s} \cup W_{y_s})] \wedge$
$[\varphi_s(\langle x_s, y_s\rangle) \in W_{x_s} \Rightarrow \varphi_s(\langle x_s, y_s\rangle) \notin B] \wedge$
$[\varphi_s(\langle x_s, y_s\rangle) \in W_{y_s} \Rightarrow \varphi_s(\langle x_s, y_s\rangle) \notin A]]]$.

Clearly, then, if A and B satisfy $\mathcal{R}_s^0$ for every s, A is r.e. inseparable from B. If A and B satisfy $\mathcal{R}_s^1$ for every s, A is *not* effectively Δ_1^0-inseparable from B.

begin *stage* $2s$;
 if $(\exists w)[w \in (W_s \cap \overline{A^{2s}})]$
  then
   let w_s be the least such w;
   put w_s into B $\{$*so that* $w_s \in (W_s \cap B)\}$
  else $\{$*hence,* W_s *is finite; therefore,* $\overline{W_s}$ *is infinite*$\}$
   let w_s be the least element of $(\overline{W_s} \cap \overline{B^{2s}})$;
   put w_s into A $\{$*so that* $w_s \in (\overline{W_s} \cap A)\}$
 endif
end $\{$*stage* $2s\}$.

begin *stage* $2s+1$;
 find x_s and y_s such that

$$W_{x_s} = \begin{cases} \{\varphi_s(\langle x_s, y_s\rangle)\}, & \text{if } \varphi_s(\langle x_s, y_s\rangle)\downarrow \notin B^{2s+1}; \\ \emptyset, & \text{otherwise;} \end{cases}$$

$$W_{y_s} = \begin{cases} \{\varphi_s(\langle x_s, y_s\rangle)\}, & \text{if } \varphi_s(\langle x_s, y_s\rangle)\downarrow \in B^{2s+1}; \\ \emptyset, & \text{otherwise;} \end{cases}$$

 if $\varphi_s(\langle x_s, y_s\rangle)\downarrow \notin (A^{2s+1} \cup B^{2s+1})$
  then
   put $\varphi_s(\langle x_s, y_s\rangle)$ into A $\{$*so that* $\varphi_s(\langle x_s, y_s\rangle)$ *can't later be put into* $B\}$
 endif
end $\{$*stage* $2s+1\}$.

It is straightforward to verify that A and B are as required. □

In recursion theory the subsets of N are usually topologized by identifying them with their characteristic functions in 2^N, placing the discrete topology on 2 ($= \{0, 1\}$), and the corresponding product topology on 2^N [Rog87, Myh61]. Similarly each pair (A, B) of *disjoint* subsets of N can be identified with a naturally corresponding function in 3^N, namely,

$$\lambda z. \begin{cases} 2, & \text{if } z \in A; \\ 1, & \text{if } z \in B; \\ 0, & \text{otherwise.} \end{cases}$$

We place the discrete topology on 3 ($= \{0,1,2\}$) and the corresponding product topology on 3^N. Now, by a theorem of Hausdorff's [Wil70, Theorem 30.3, Page 216], both these product topologies are homeomorphic to the *Cantor-set* topology, a complete, metrizable space; hence, the product topologies satisfy Baire's Theorem.

Corollary 1 just below says that the pairs (A, B) witnessing the truth of Theorem 6 just above are plentiful in the sense of Baire Category.

Corollary 1 $\{(A, B) \mid (A, B)$ *witnesses the truth of Theorem 6*$\}$ *is co-meager.*

Proof. At any stage in the construction of the proof of Theorem 6 a finite number of elements have been committed to each of A, B, $\overline{A}$, and $\overline{B}$. Modify this construction so that, for each $i \in \{0,1\}$, every reference to $2s+i$ is changed to $3s+i+1$ and an "opponent" in a new *stage* $3s$ is allowed to commit a finite number of elements to A, B, $\overline{A}$, and $\overline{B}$, provided A is kept disjoint from B and each stage honors the commitments of the previous. The A (respectively, B) resulting from this modified construction is understood to be the set of numbers explicitly committed to A (respectively, B). Clearly the A and B from the modified construction *still* satisfy Theorem 6. By the standard connection between infinite games and Baire Category (first noticed by Banach [Jec78]), we have Corollary 1. $\square$

The sets A and B constructed in the proof of Theorem 6 are clearly recursive in the halting problem (*Note.* In that construction $A = \overline{B}$). Of course A cannot be r.e. However, if all we wanted was to show that effective inseparability and effective Δ^0_1-inseparability are not coextensive, conceivably A and B could both be r.e. Theorem 7 just below implies they could not. We did not explore whether, in Theorem 6, B can be r.e. and A recursive in the halting problem. We also did not investigate whether a measure-theoretic [Rog87] analog of Corollary 1 holds.

Theorem 7 *Suppose A and B are r.e. Then: A is effectively inseparable from B $\Leftrightarrow$ A is effectively Δ^0_1-inseparable from B.*

Proof. Suppose A and B are disjoint r.e. sets. It clearly suffices to show [A is effectively inseparable from $B \Rightarrow A$ is effectively Δ^0_1-inseparable from B]. Suppose A is effectively inseparable from B as witnessed by the recursive function f. By the Double Recursion Theorem there are recursive functions g and h such that, for all x and y,

$$W_{g(x,y)} = \begin{cases} A \cup \{f(g(x,y), h(x,y))\}, & \text{if } f(g(x,y), h(x,y)) \text{ appears in } W_x \text{ before } W_y; \\ A, & \text{otherwise;} \end{cases}$$

$$W_{h(x,y)} = \begin{cases} B \cup \{f(g(x,y), h(x,y))\}, & \text{if } f(g(x,y), h(x,y)) \text{ appears in } W_y \text{ before (or} \\ & \quad \text{at the same time as it appears in) } W_x; \\ B, & \text{otherwise.} \end{cases}$$

We show that A is effectively Δ^0_1-inseparable from B as witnessed by $\lambda x, y \,.\, f(g(x,y), h(x,y))$.

Case (1). $f(g(x,y), h(x,y))$ appears in W_x before W_y. Then

$$W_{g(x,y)} = A \cup \{f(g(x,y),h(x,y))\} \text{ and } W_{h(x,y)} = B.$$

Suppose for contradiction that $f(g(x,y),h(x,y)) \notin B$. Then $A \subseteq W_{g(x,y)} \subseteq \overline{W_{h(x,y)}} \subseteq \overline{B}$. Therefore, by Definition 2, $f(g(x,y),h(x,y)) \in (\overline{W_{g(x,y)}} \cap \overline{W_{h(x,y)}})$. This contradicts that $W_{g(x,y)} = A \cup \{f(g(x,y),h(x,y))\}$. Hence, $f(g(x,y),h(x,y)) \in (W_x \cap B)$.

Case (2). $f(g(x,y),h(x,y))$ appears in W_y before (or at the same time as it appears in) W_x. Then, by an argument symmetric to that of Case (1), $f(g(x,y),h(x,y)) \in (W_y \cap A)$.

Case (3). Neither Cases (1) nor (2). Then, $f(g(x,y),h(x,y)) \in (\overline{W_x} \cap \overline{W_y})$. □

We indicate, briefly, how to generalize up into the arithmetical hierarchy all the inseparability notions above. We let φ^n denote a fixed acceptable programming system (numbering) for the set of partial functions: $N \to N$ which are partial recursive in $\emptyset^{(n)}$ [LMF76, Rog87]. We let W_p^n denote the domain of φ_p^n. W_p^n is, then, the Σ_{n+1}^0 set ($\subseteq N$) accepted by φ^n-program p. We present the definition of *effective* Δ_{n+1}^0*-inseparability* to illustrate the pattern of the generalizations. The other inseparability notions above are similarly generalized.

Definition 4 *A is* effectively Δ_{n+1}^0-inseparable from $B \Leftrightarrow [(A \cap B) = \emptyset \wedge (\exists$ *recursive* $f)(\forall x,y)[f(x,y) \in ((W_x^n \cap B) \cup (\overline{W_x^n} \cap A) \cup (W_y^n \cap A) \cup (\overline{W_y^n} \cap B) \cup (W_x^n \bigtriangleup \overline{W_y^n}))]]$.

Since the inseparability concepts defined above are clearly invariant under choice of acceptable programming system, we have, for example, that the $n = 0$ case of Definition 4 agrees with Definition 1.

It is clear that by employing the appropriate relativization [Rog87, LMF76] of recursion theoretic tools each of our results above *except* Theorems 3 and 4 and Remarks 1 and 2 can be generalized to each higher level of the arithmetical hierarchy. Theorems 3 and 4 provide sufficient conditions for effective Δ_1^0-inseparability (respectively, recursive inseparability) of *subrecursive* index sets. Theorem 8 below provides a sufficient condition for effective Δ_2^0-inseparability of subrecursive index sets. We have not explored any other levels for sufficient conditions involving inseparability of subrecursive index sets.

We note that Royer and Case [RC89] use effective Δ_2^0-inseparability to gain insight into theorems of Ladner, Schöning and Ambos-Spies [Lad75, Sch82, AS85] in structural complexity theory and to obtain independence results about complexity. These latter results are in the style of independence results due to Regan, Kowalczyk, Hartmanis, and Kurtz, O'Donnell, and Royer [Reg83, Kow84, Har85, Reg86, Reg87, KOR87].

Theorem 8 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are disjoint subsets of* $\mathbf{S}$. *Suppose there are ψ-programs c and d such that, for every* finite *set A,* $(\psi_d|_A \cup \psi_c|_{\overline{A}}) \in \mathbf{C}$ *and* $(\psi_c|_A \cup \psi_d|_{\overline{A}}) \in \mathbf{D}$. *Then: $\mathcal{S}_{\mathbf{C}}$ is effectively Δ_2^0-inseparable from $\mathcal{S}_{\mathbf{D}}$.*

The quantifier '$\forall^\infty$' from [Blu67] means 'for all but finitely many'. If we suppose, as for the examples following the statement of Theorem 3 above, that ψ is a subrecursive programming system for $\mathbf{S} = \mathcal{C}$ubictime and that it satisfies Assumption 2, then, by Theorem 8, the first example after the statement of Theorem 3 also provides an example of effective

Δ_2^0-inseparability. It is easy to argue that each of the *other* two example pairs of sets after the statement of Theorem 3 has a Δ_2^0 separating set. However, by Theorem 8 once again, $\{r \mid \mathrm{range}(\psi_r) \subseteq \{0,1\} \wedge (\forall^\infty z)[\psi_r(z) = 0]\}$ is effectively Δ_2^0-inseparable from $\{r \mid \mathrm{range}(\psi_r) \subseteq \{0,1\} \wedge (\forall^\infty z)[\psi_r(z) = 1]\}$.

PROOF OF THEOREM 8. Suppose the hypotheses.

Since effective Δ_2^0-inseparability is invariant under choice of acceptable programming system, we are free to judiciously choose one. Now, the functions recursive in $\emptyset^{(1)}$ are well known to coincide with the functions which are the limit of some recursive function [Sho59, Sho71, Soa87]. This and its relativization were first noticed and used by Post [Sha71] and have been employed (sometimes with rediscovery) many times. Case [Cas83] exploited an extension to partial functions. Meyer [Mey72] proved a lemma that easily generalizes to the fact that each function recursive in $\emptyset^{(1)}$ is the limit of some *primitive* recursive function. Royer and Case [RC86, RC89] construct an $\mathcal{L}$ineartime function L_* such that, if we define

$$\varphi_p^* = \lambda z \centerdot \lim_{t\to\infty} L_*(p, z, t),$$

then φ^* is an acceptable programming system (numbering) for the set of partial functions partial recursive in $\emptyset^{(1)}$. They further define a particular associated *relativized* Blum complexity measure [LMF76], Φ^*, based on the modulus of convergence [Sho59, Sho71] for L_*. In particular the predicate $\lambda p, z, t \centerdot [\Phi_p^*(z) \leq t]$ is in Π_1^0 (generalizing Shoenfield's Modulus Lemma [Sho59, Sho71, Soa87]); hence, this predicate is in Δ_2^0. Let W_p^* denote the domain of φ_p^*. W_p^* is, then, the Σ_2^0 set ($\subseteq N$) accepted by φ^*-program p. We say that a number w *appears in* W_x^* *before* W_y^* $\Leftrightarrow \Phi_x^*(w)\downarrow < \Phi_y^*(w) \leq \infty$. A number w *appears in* W_x^* *at the same time as it appears in* W_y^* $\Leftrightarrow \Phi_x^*(w) = \Phi_y^*(w)\downarrow < \infty$.

The Hybrid Recursion Theorem of Royer and Case [RC86, RC89] holds between ψ and φ^* [RC86, RC89] and can be used to obtain self-other reference and facilitate interaction between these systems. By the Hybrid Recursion Theorem, then, there are $\mathcal{L}$ineartime functions f and g such that, for all x, y, and z,

$$\psi_{f(x,y)}(z) = \begin{cases} \psi_d(z), & \text{if } L_*(g(x,y), 0, z) = 0; \\ \psi_c(z), & \text{if } L_*(g(x,y), 0, z) > 0; \end{cases}$$

$$\varphi^*_{g(x,y)}(z) = \begin{cases} 0, & \text{if } f(x,y) \text{ appears in } W_x^* \text{ before } W_y^*; \\ 1, & \text{if } f(x,y) \text{ appears in } W_y^* \text{ before (or at} \\ & \quad \text{the same time as it appears in) } W_x^*; \\ \uparrow, & \text{otherwise.} \end{cases}$$

(In this application of the Hybrid Recursion Theorem there is no direct self-reference, but there is circular reference since, for each x and y, ψ-program $f(x,y)$ and φ^*-program $g(x,y)$ each refers to the other.) It suffices to show that

$$(\forall x, y)[f(x,y) \in ((W_x^* \cap \mathcal{S}_{\mathbf{D}}) \cup (W_y^* \cap \mathcal{S}_{\mathbf{C}}) \cup (\overline{W_x^*} \cap \overline{W_y^*}))].$$

Case (1). $f(x,y)$ appears in W_x^* before W_y^*. Then

$$\varphi^*_{g(x,y)}(0) = 0 = \lim_{z\to\infty} L_*(g(x,y), 0, z).$$

Hence, $A = \{z \mid L_*(g(x,y),0,z) > 0\}$ is a *finite* set. Therefore, $\psi_{f(x,y)} = (\psi_c|_A \cup \psi_d|_{\overline{A}})$, which, by hypothesis, is in $\mathbf{D}$. Hence, $f(x,y) \in \mathcal{S}_{\mathbf{D}}$. Therefore, $f(x,y) \in (W_x^* \cap \mathcal{S}_{\mathbf{D}})$.

Case (2). $f(x,y)$ appears in W_y^* before (or at the same time as it appears in) W_x^*. Then, by an argument symmetric to that of Case (1), $f(x,y) \in (W_y^* \cap \mathcal{S}_{\mathbf{C}})$.

Case (3). Neither Cases (1) nor (2). Then, $f(x,y) \in (\overline{W_x^*} \cap \overline{W_y^*})$. □

Remark 3 *The sufficient condition of Theorem 8 is not necessary.*

Proof. We let $\mathbf{C} = \{r \mid \mathrm{range}(\psi_{\psi_r(0)}) \subseteq \{0,1\} \wedge (\forall^\infty z)[\psi_{\psi_r(0)}(z) = 0]\}$ and $\mathbf{D} = \{r \mid \mathrm{range}(\psi_{\psi_r(0)}) \subseteq \{0,1\} \wedge (\forall^\infty z)[\psi_{\psi_r(0)}(z) = 1]\}$. $\mathbf{C}$ and $\mathbf{D}$ are non-trivial since $\mathbf{S}$ contains $\mathcal{L}$ineartime, and, clearly, $\mathbf{C}$ and $\mathbf{D}$ are disjoint, but do not satisfy the hypotheses of Theorem 8. Nonetheless, we show, employing terminology and results from the proof of Theorem 8, that $\mathcal{S}_{\mathbf{C}}$ is effectively Δ_2^0-inseparable from $\mathcal{S}_{\mathbf{D}}$. The techniques of [RC86, RC89] easily establish various delayed forms [Cas74] of the Hybrid Recursion Theorem [RC86, RC89]. In particular, there are $\mathcal{L}$ineartime functions f and g such that, for all w, x, y, and z,

$$\psi_{\psi_{f(x,y)}(w)}(z) = \begin{cases} 1, & \text{if } L_*(g(x,y),0,z) = 0; \\ 0, & \text{if } L_*(g(x,y),0,z) > 0; \end{cases}$$

$$\varphi^*_{g(x,y)}(z) = \begin{cases} 0, & \text{if } f(x,y) \text{ appears in } W_x^* \text{ before } W_y^*; \\ 1, & \text{if } f(x,y) \text{ appears in } W_y^* \text{ before (or at} \\ & \quad \text{the same time as it appears in) } W_x^*; \\ \uparrow, & \text{otherwise.} \end{cases}$$

The rest of the proof is a straightforward modification of the proof of Theorem 8. □

We have not explored whether the proofs of Theorems 3 and 4 and Remarks 1 and 3 might, upon suitable generalization, lead to characterization theorems. Perhaps algebraic manipulations in the style of [Str68, Wag69] (see also [Fri71, Bye82a, Bye82b, Bye83, Bye84b, Bye84a, Iva86]), but for *subrecursive* systems, would be useful to *elegantly* bring the sufficient, but not necessary conditions of this paper closer to characterizations.

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