A1 **Logic** (25 points)

For each of the following, either prove it to be correct using resolution or provide an explicit interpretation witnessing it to be incorrect.

a. (6.25 points)

$\models [(\exists x)[P(x) \lor Q(x)] \rightarrow [(\exists x)P(x) \lor (\exists x)Q(x)]]$

b. (6.25 points)

$\{(\forall x)[P(g(b,x)) \rightarrow \neg Q(b)], (\forall x)[P(x) \rightarrow (\forall y)Q(y)]\} \models (\exists x)\neg P(x)$

c. (6.25 points)

$\{(\forall x)(\exists y)P(x,y), (\forall x)(\exists y)P(y,x)\} \models (\exists x)P(x,x)$

d. (6.25 points)

$\{(\forall x)[P(x) \rightarrow S(f(x))], (\forall x)[P(x) \rightarrow R(x,f(x))], P(a), (\forall x)[R(a,x) \rightarrow T(x)]\}$
$\models (\exists x)[T(x) \land S(x)]$

A2 **Logic** (25 points)

For each $n \geq 1$, let α_n be the closed wff

$$(\exists x_1)\ldots(\exists x_n)[\wedge_{i\neq j}\neg(x_i = x_j)]$$

(a) (5 points) What can you say about any normal interpretation[1] in which α_3 is true and one in which it is false.

(b) (15 points) Prove that there is no set of closed wffs which has arbitrarily large finite normal models but has no infinite normal model.

Hint. BWOC suppose there is a set of closed wffs, Γ, which has arbitrarily large finite normal models and but has no infinite normal model. Consider $\Gamma' = \Gamma \cup \{\alpha_n \mid n \geq 1\}$. Show it is satisfiable (you may use compactness theorem). You can reach the desired contradiction by considering the models of Γ'.

(c) (5 points) Assume a first-order language (with equality) with a binary predicate symbol R. Give a closed wff that is true in some infinite normal interpretation but is not true in all finite normal interpretations.

[1]A normal interpretation is one in which the binary predicate symbol '=' is interpreted as the actual equality relation on the universe (domain) of discourse.

A3 **Logic** (25 points)

Let $\mathcal{I}$ be an interpretation (or structure) for a predicate logic formula F with corresponding universe of discourse $U_{\mathcal{I}}$. For *example*: for each variable x in F, $\mathcal{I}(x) \in U_{\mathcal{I}}$; for each n-ary predicate symbol P in F, $\mathcal{I}(P) \subseteq U_{\mathcal{I}}^n$; for each n-ary function symbol f in F, $\mathcal{I}(f) : U_{\mathcal{I}}^n \rightarrow U_{\mathcal{I}}$; ... .

Suppose s, t are terms built from the constituents of F and x is a variable from F and that G is a formula built from the constituents of F.

Then we write

$$\{s\}[x/t] \tag{1}$$

to mean the result of simultaneously substituting for each occurrence of x in the term s, the term t.

We write

$$\{G\}[x/t] \tag{2}$$

to mean the result of simultaneously substituting for each *free* occurrence of x in the formula G, the term t.

Suppose $u \in U_{\mathcal{I}}$.

Then we write

$$\mathcal{I}[x/u] \tag{3}$$

to mean the variant of the interpretation $\mathcal{I}$ which is just like $\mathcal{I}$ *except* that $\mathcal{I}[x/u]$ interprets x to mean u, i.e., $\mathcal{I}[x/u](x) \stackrel{\text{def}}{=} u$; whereas, $\mathcal{I}(x)$ may or may not $= u$.

a. (6.25 points)

Prove by mathematical induction on the logical complexity of the term s that

$$\mathcal{I}[x/\mathcal{I}(t)](s) = \mathcal{I}(\{s\}[x/t]). \tag{4}$$

b. (12.50 points)

N.B. For *this* part of A3 you may use without proof the result from A3 part a.

Definition t *is free for* x *in* F $\stackrel{\text{def}}{\Leftrightarrow}$ for *no* variable y in t does x occur free within any subformula F' of F, *where* F' is either of the form $(\forall y)H$ or of the form $(\exists y)H$.

Example The term $f(x, y)$ *is* free for x in the formula $(\exists z)P(x, z)$.

Example The term $f(x, y)$ is *not* free for x in the formula $(\exists y)P(x, y)$.

Suppose t *is free for* x *in* F. Then: *prove by mathematical induction* on the logical complexity of the formula F that

$$\mathcal{I}[x/\mathcal{I}(t)](F) = \mathcal{I}(\{F\}[x/t]). \tag{5}$$

N.B. *Be sure that your proof makes it very clear how you are using the hypothesis that* t *is free for* x *in* F.

c. (6.25 points)

Explicitly present an example t, x, and F for which *both*

- t is *not* free for x in F *and*
- (5) above *fails.*

N.B. *You need not show your* t, x, *and* F *work, just make sure they do.*

A4 **Logic** (25 points)

Here is one form of the *Completeness Theorem* for first order predicate logics, and you may and should use *this* form with*out* proof in *this* problem.[2]

Theorem 1 (Completeness) Suppose ℓ is a first order predicate logic language. Suppose $(\Gamma \cup \{A\})$ is a set of formulas of ℓ. $\vdash$ is the *provability relation* for a fixed sound and complete system of proof for ℓ.[3] $\models$ is the (semantic) *consequence* relation for ℓ.[4]

Then: $\Gamma \models A \Leftrightarrow \Gamma \vdash A$.

Here is one form of the *Compactness Theorem* for (first order) predicate logics,

Theorem 2 (Compactness) Suppose ℓ is a first order predicate logic language. Suppose Γ is a set of formulas of ℓ.

Then: Γ has a model $\Leftrightarrow (\forall\ \textit{finite}\ \Delta \subseteq \Gamma)[\Delta$ has a model$]$.

Prove the above form of the Compactness Theorem from the above form of the Completeness Theorem employing the Hint just below.

Hint: Assume the above form of the Completeness Theorem.

Of course, do *not* assume Γ is finite. (◡̈)

The $(\Rightarrow)$ direction of Compactness is easy without even explicitly employing the Completeness assumption.

For the $(\Leftarrow)$ direction of Compactness, prove the contrapositive, i.e., prove the negation of the left-hand side implies the negation of the right-hand side. First assume Γ has *no* model. Suppose B is a fixed closed formula. Show that $\Gamma \models (B \wedge \neg B)$.

Employ *each direction* of the Completeness Theorem **and something about the SIZE of proofs (in the fixed system of proof)** to get a *finite* $\Delta \subseteq \Gamma$ such that $\Delta \models (B \wedge \neg B)$ *too*. Then show that *this* Δ has no model.

---

[2]IF you want to use a different form, you will have to state *and* prove it.

[3]This system can be, for *example*, a resolution system, one of many tableaux systems, a Hilbert style system, a Gentzen style system, ...

$\Gamma \vdash A$ means that A is *provable from* Γ in the fixed system of proof.

[4]$\Gamma \models A$ means that every model of Γ satisfies A.