A1. **Logic** (25 points)

a. (5 points)
Disprove by exhibiting a suitable structure or interpretation.

$$\{F_1, F_2\} \models F_3$$

where

$F_1 = (\forall x)(\forall y)[R(x,y) \rightarrow \neg R(y,x)]$,
$F_2 = (\forall x)(\exists y)[R(x,y)]$,
$F_3 = (\forall x)(\forall y)(\forall z)[[R(x,y) \wedge R(y,z)] \rightarrow R(x,z)]$.

b. (10 points)
Prove step by step using resolution. Show all of your work.

$$\{F_1, F_2, F_3\} \models F_4$$

where

$F_1 = (\exists x)[T(x) \wedge P(x) \wedge (\forall y)[R(x,y) \rightarrow T(y)]]$,
$F_2 = (\forall x)[P(x) \rightarrow [Q(x) \vee (\exists y)[S(y) \wedge R(x,y)]]]$,
$F_3 = (\forall x)[T(x) \rightarrow \neg S(x)]$,
$F_4 = (\exists x)[Q(x)]$.

c. (10 points)
Prove step by step using resolution. Show all of your work.

$$\{F_1, F_2, F_3, F_4\} \models F_5 \ .$$

where

$F_1 = (\forall x)(\forall y)(\forall z)[[P(x,y) \wedge Q(z,x) \wedge R(z)] \rightarrow S(y,z)]$,
$F_2 = (\forall x)(\exists y)[R(x) \rightarrow [T(y) \wedge Q(x,y)]]$,
$F_3 = (\forall x)(\exists y)[T(x) \rightarrow P(x,y)]$,
$F_4 = (\exists x)[R(x)]$,
$F_5 = (\exists x)(\exists y)[S(x,y)]$.

A2 **Logic** (25 points) (**Propositional Logic**)

a. (15points)

Let S be a satisfiable set of wffs. Let $F_0, F_1, F_2, \ldots$ be an enumeration all wffs. Consider the following sequence:

$$S_0 = S$$
$$S_{i+1} = \begin{cases} S_i \bigcup \{F_i\} & \text{if } S_i \bigcup \{F_i\} \text{ is satisfiable} \\ S_i & \text{otherwise} \end{cases}$$

Let $\Delta = \bigcup_{n \geq 0} S_n$. Show that

i. (3 points) Each S_n is satisfiable, $n \geq 0$.
ii. (6 points) For each wf F, $[F \in \Delta \text{ iff } \neg F \notin \Delta]$
iii. (6 points) For all wffs F, G, $[(F \vee G) \in \Delta \text{ iff } F \in \Delta \text{ or } G \in \Delta]$

b. (10 points)

For this question, do not assume that the compactness theorem has been established.
Let S be a set of wff and F be a wff. Now, consider the following two statements:

(I) If $S \models F$, then for some finite subset, $S_f \subseteq S$, $S_f \models F$.
(II) If every finite subset of S is satisfiable then S is satisfiable.

i. (5 points) Show that if (I) holds then (II) holds.
ii. (5 points) Show that if (II) holds then (I) holds.

For this question you may assume that $S \models F$ iff $S \bigcup \{\neg F\}$ is unsatisfiable.

A3. **Logic** (25 points)

Here is one form of the *Compactness Theorem* for first order predicate logics.

Theorem (Compactness) Suppose L is a first order predicate logic language. Suppose Γ is a set of formulas of L. Then Γ has a model if and only if every finite subset of Γ has a model.

In *this* problem, you should *explicitly employ* this form of the Compactness Theorem without proof.

Recall that graphs are defined by specifying a set of nodes and a set of edges or arcs connecting nodes. Each edge or arc connects two nodes (which may or may not be the same node).

Let L be a first order predicate logic language containing at least one binary predicate R and at least the constants a and b.

a. (5 points) Explain how language L can be used to specify a graph.

b. (5 points) For each natural number $n > 0$, define a formula $path_n(x,y)$ in the language L that will be true exactly when there is a path of length n from node x to node y in a graph.

c. (15 points) Using the Compactness Theorem (above), prove that it is not possible to define in L the general notion of a finite path from one node in the graph to another node. Hint: Suppose that there is a formula $path(x,y)$ that will be true exactly when there is a path (of any finite length) from node x to node y. Consider the set of formulas $\{\neg path_n(a,b) \mid n > 0\} \cup \{path(a,b)\}$.

A4 **Logic** (25 points)

Here is one form of the *Compactness Theorem* for (first order) predicate logics.

Theorem (Compactness) Suppose ℓ is a first order predicate logic language. Suppose Γ is a set of formulas of ℓ.

Then: Γ has a model $\Leftrightarrow (\forall$ *finite* $\Delta \subseteq \Gamma)[\Delta$ has a model$]$.

In *this* problem, you may and should *explicitly employ* this form of the Compactness Theorem with*out* proof.

Definition

1. $(U, <)$ is a *partial ordering on* $U \overset{\text{def}}{\Leftrightarrow} U$ is a set and $<$ is a binary relation on U such that

$$(\forall u, v \in U)[u < v \Rightarrow v \not< u] \tag{1}$$

   and

$$(\forall u, v, w \in U)[[u < v \text{ and } v < w] \Rightarrow u < w]. \tag{2}$$

2. $(U, <)$ is a *total ordering on* $U \overset{\text{def}}{\Leftrightarrow} (U, <)$ is a partial ordering on U and

$$(\forall u, v \in U \mid u \neq v)[u < v \text{ or } v < u]. \tag{3}$$

Examples

1. Suppose U is the set of nodes, $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in the finite graph depicted in Figure 1 below and $<$ on U is the transitive closure of the (directed) edge relation depicted by arrows in this figure. For *example*, in Figure 1, we have $5 < 2$.
   This $(U, <)$ *is* a total ordering on U.

```
5 --> 3 --> 6 --> 8 --> 7 --> 4 --> 1 --> 2
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Figure 1: Example 1

2. Suppose U is the set of nodes, $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in the finite graph depicted in Figure 2 below and $<$ on U is the transitive closure of the (directed) edge relation depicted by arrows in this figure. For *example*, in Figure 2, we have $5 < 1$. This $(U, <)$ is partial ordering on U but *not* a total ordering on U. It *does* satisfy (1) and (2) above; however, it does *not* satisfy (3). For *example*, in Figure 2, we have $1 \not< 2$ and $2 \not> 1$, and, also, $4 \not< 7$ and $7 \not> 4$.
   (Problem A4 continues onto the next page.)

```
        8 --> 2

5 --> 3 --> 4
      |
       ---> 7
            |
            V
      6 --> 1
```

Figure 2: Example 2

(This page is the continuation of Problem A4.)

3. Suppose U is the set of all real numbers and $<$ on U stands for the ordinary less-than relation on U. This $(U,<)$ *is* a total ordering on U.
4. Suppose U is the *set of all subsets of* the set of real numbers. Then $(U,\subset)$ is a partial ordering on U. You may find it hard to think of an example $<$ so that $(U,<)$ is a *total order* on U.

Definition Suppose $(U,<)$ and $(U,<')$ are each partial orderings on U.

$(U,<')$ *extends* $(U,<)$ $\overset{\text{def}}{\Leftrightarrow}$

$$(\forall u,v \in U)[u < v \Rightarrow u <' v]. \tag{4}$$

E.g., Example 1 above extends Example 2 above.

Explicitly apply the form of the Compactness Theorem above and employ the Hint below to prove

$$(\forall \text{ partial orderings } (U,<))(\exists <')[(U,<') \text{ is a total ordering on } U \text{ which extends } (U,<)]. \tag{5}$$

Hint: Suppose $(U,<)$ is a partial ordering on U.

Define a first order predicate logic language ℓ and a set of formulas Γ of ℓ such that

(a) ℓ contains a constant symbol c_u for each $u \in U$ and a binary predicate symbol $<$; and

(b) Γ contains *two* axioms expressing (1) and (2) above, respectively, *and all the axioms of the two forms*

$$c_u < c_v, \tag{6}$$

where $u, v \in U$ *and* $u < v$, *and*

$$[c_u < c_v \ \vee \ c_v < c_u], \tag{7}$$

where $u, v \in U$ *and* $u \neq v$.

Suppose V is any *finite* subset of U. Show that, for the partial ordering (V, $<$ restricted to V), there is a total ordering $(V,<')$ extending it. To do this, suppose $V = \{v_1,\dots,v_n\}$, $n \geq 0$. Pick out the $<$-minimal elements of V and order them by their v-subscripts. Pull them out of V and iterate until no longer possible. String the results of each iteration after one another to (graphically) construct $<'$.[1] *Make this construction more detailed and convincing.*

Suppose Δ is any *finite* subset of Γ. Consider *which* constant symbols c_u are explicitly mentioned in this Δ. Obtain a model for Δ.

Next apply the form of the Compactness Theorem above to obtain a model of Γ.

Consider:

- the interpretation *in this latter model* of each constant symbol c_u, for $u \in U$, *and*
- the interpretation *in this latter model* of the binary predicate symbol $<$.

[1]For Example 2 above the $<$-minimal elements of *that* U are 8, 5, 6. Following the iterative procedure on *this* example will yield a total ordering on this U extending $(U,<)$ different from the one in Example 1.