C1 **Theory** (25 points)

a. (6.25 points)
Let $L = \{x \in \{a,b\} \mid x$'s final five symbols include two a's and three b's$\}$. *Explicitly prove by Myhill-Nerode* that L *is* regular.

b. (6.25 points)
Let $L = \{w \cdot w^R \mid w \in \{a,b\}^*\}$, *where* w^R is w spelled backwards, and '$\cdot$' is the string concatenation operation — not an alphabet symbol. *Explicitly prove by Generalized Pumping for Finite Automata* that L is *not* regular.

c. (6.25 points)
Construct a *four* state finite automaton $\mathcal{M}$ for accepting $L = \{ab\}$ *and explicitly prove by Myhill-Nerode* your $\mathcal{M}$ *is* minimal state.

d. (6.25 points)
Explicitly use Generalized Pumping for Finite Automata to show that *no* finite automaton accepting $L = \{ab\}$ has fewer than *three* states.

C2 **Theory** (25 points)

Let $L_{wwr} \overset{\text{def}}{=} \{w \cdot w^R \mid w \in \{a,b\}^*\}$, where w^R is w spelled backwards, and the $\cdot$ denotes string concatenation not an alphabet symbol.

a. (5.0 points)

Explicitly draw the state diagram of a PDA for accepting L_{wwr}.

b. (10.0 points)

Show that the language (over the alphabet $\{a,b\}$) $\overline{L_{wwr}}$ is *also* a CFL.

Hint: First *show*

$$\overline{L_{wwr}} = (L_{odd} \cup L'), \tag{1}$$

where

$$L_{odd} = \{x \in \{a,b\}^* \mid |x| \text{ is odd}\}, \tag{2}$$

and

$$L' = \{uv \in \{a,b\}^* \mid |u| = |v| \ \wedge \ u \neq v^R\}. \tag{3}$$

Then *show step-by-step the relevance of* (1) above.

c. (10.0 points)

Let $L = \{ab^{n^2} \mid n \geq 0\}$. *Explicitly employ Pumping for PDA to show* that L is *not* a CFL.

C3 **Theory** (25 points)

Let N denote the set of non-negative integers.

Definition *Consider all the finite sets of equations defining primitive recursive functions and which contain a special one argument function letter* $\mathbf{f}$. *Gödel number (code number) 1-1 onto* N *all these finite sets of equations.*

1. *Let* E_q *be (by definition) the finite set of equations with Gödel number* q.
2. $f_q \stackrel{\text{def}}{=}$ *the primitive recursive function which* $\mathbf{f}$ *defines in* E_q.

Clearly, then, $f_0, f_1, f_2, \ldots$ is a list of all and only the primitive recursive functions of one argument. You may and should use the following theorem with*out* proof.

Theorem *For each* $x \in N$, *let*

$$g(x) = 1 + f_x(x). \tag{4}$$

Then g *is computable, but* not *primitive recursive.*

Explicitly use the Hint just below to prove the following

Corollary 1 $\lambda x.\, f_x(x)$ *is also computable, but* not *primitive recursive.*

You may also use with*out* proof the primitive recursiveness of functions in standard lists of primitive recursive functions. You must *say* when you are using one of these!

N.B. Do *not* prove the theorem just above.

Hint: Show that $\lambda x.\, f_x(x)$ is computable *using the Theorem.*

To show that $\lambda x.\, f_x(x)$ is not primitive recursive, suppose for contradiction otherwise and then *use the Theorem.*

C4 **Theory** (25 points)

Fix a standard programming formalism φ for computing all the *one-argument* partial computable functions which map the non-negative integers into themselves. Code (Gödel) number the φ-programs *onto* the entire set of non-negative integers. Let φ_p denote the partial function computed by program (number) p in the φ-system. Let Φ denote a standard Blum step-counting measure associated with φ.[1] Let $W_p \stackrel{\text{def}}{=}$ the domain of φ_p.[2] You may assume *without* proof that in the φ-system Universality, S-m-n, *and the Kleene Recursion Theorem (KRT)* hold.

The first two parts of this question will lead you (with *very* useful hints) through a proof of the following

Theorem *Suppose* Δ is a collection of r.e. sets. *Let*

$$P_\Delta \stackrel{\text{def}}{=} \{p \mid W_p \in \Delta\}. \tag{5}$$

Suppose P_Δ is r.e.
Then

$$(\forall p)[W_p \in \Delta \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta]]. \tag{6}$$

The third and fourth parts of this question each asks you to apply the theorem and also provides very useful hints. In that interest and for later use, let

$$A = \{p \mid W_p = \{0\}\}. \tag{7}$$

a. (6.25 points)

Assume all the hypotheses of the Theorem. *Explicitly use KRT* in the φ-system (formally or informally — as you choose) to prove that

$$(\forall p)[W_p \in \Delta \Rightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta]]. \tag{8}$$

Hint for C4(a)**:** Suppose that $W_p \in \Delta$. Suppose for contradiction that ($\forall$ finite sets $D \subseteq W_p)[D \notin \Delta]$. Apply KRT to obtain an self-referential e which determines its I/O behavior on input x *in part* according to whether or not "e appears in P_Δ within x steps." Make this precise, figure out what to have e do in each case, etc., and get a contradiction.

b. (6.25 points)

Assume all the hypotheses of the Theorem. *Explicitly use KRT* in the φ-system (formally or informally — as you choose) to prove that

$$(\forall p)[(\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta] \Rightarrow W_p \in \Delta]. \tag{9}$$

Hint for C4(b)**:** Suppose ($\exists$ a finite set $D \subseteq W_p)[D \in \Delta]$. Let D be an example. Suppose for contradiction that $W_p \notin \Delta$. Apply KRT to obtain an self-referential e which determines its I/O behavior on input x *in part* according to whether it eventually discovers that "[$x \in D \vee e$ appears in P_Δ]." Make this precise, figure out what to have e do if it makes this discovery, *etc.*

c. (6.25 points)

Explicitly use the Theorem stated above in this question, C4, to show that A is *not* r.e., *where* A is defined in (6) above.

Hint for C4(c)**:** Suppose for contradiction otherwise. Clearly $A = P_\Delta$ for $\Delta = \{\{0\}\}$. Therefore, from (5) above, we have that $(\forall p)[W_p = \{0\} \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D = \{0\}]]$. Pick D and W_p so that $D = \{0\} \subseteq W_p \neq \{0\}$. Get a contradiction.

d. (6.25 points)

Explicitly use the Theorem stated above in this question, C4, to show that $\overline{A}$ is *not* r.e., *where* A is defined in (6) above.

Hint for C4(d)**:** Suppose for contradiction otherwise. Clearly $\overline{A} = P_\Delta$ for $\Delta = \{W_p \mid W_p \neq \{0\}\}$. Therefore, from (5) above, we have that $(\forall p)[W_p \neq \{0\} \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \neq \{0\}]]$. Pick a D and a W_p to get a contradiction.

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[1] Hence, (i) $(\forall p)[\text{domain}(\Phi_p) = \text{domain}(\varphi_p)]$, *and* (ii) $[\{(p, x, t) \mid \Phi_p(x) \leq t\}$ is an algorithmically decidable set].

[2] Then $W_0, W_1, W_2, \ldots$ provides a standard listing of *all* the r.e. sets (of non-negative integers).