A1 **Logic** (25 points)

For each of the following *either* prove it step-by-step using resolution or another sound and complete technique of your *stated* choice *or* disprove it by exhibiting *in detail* a relevant model.

For each case of a disproof, try nonetheless to prove the result *by resolution and explicitly and clearly explain* exactly how you get blocked in trying to derive the empty clause.

Hint for A1**:** Exactly one of the four is *disprovable.*

Again: for the one disproof (besides exhibiting a relevant model) *also* try to prove the result by resolution *and explicitly and clearly explain* exactly how you get blocked in trying to derive the empty clause.

a. (6.25 points)

$$\{G_1, G_2, G_3\} \models G_4, \tag{1}$$

where we let

$$G_1 = (x)(y)[R(x,y) \supset \sim R(y,x)],^1 \tag{2}$$

$$G_2 = (x)(y)(z)[[R(x,y) \ \wedge \ R(y,z)] \supset R(x,z)], \tag{3}$$

$$G_3 = (x)(\mathsf{E}y)R(x,y), \ \textit{and} \tag{4}$$

$$G_4 = \sim(\mathsf{E}x)(y)R(y,x). \tag{5}$$

b. (6.25 points)

$$\models (x)[(\mathsf{E}y)R(x,y) \supset (\mathsf{E}y)R(y,y)]. \tag{6}$$

c. (6.25 points)

$$\{(x)[B(x) \supset A(x)]\} \models (x)[(\mathsf{E}y)[B(y) \ \wedge \ H(x,y)] \supset (\mathsf{E}y)[A(y) \ \wedge \ H(x,y)]]. \tag{7}$$

d. (6.25 points)

$$\{B_1, B_2, B_3\} \models B_4, \tag{8}$$

where we let

$$B_1 = (x)(y)[H(y) \supset R(x,y)], \tag{9}$$

$$B_2 = (x)(y)[F(x) \supset R(x,y)], \tag{10}$$

$$B_3 = (x)[\sim H(x) \supset F(x)], \ \textit{and} \tag{11}$$

$$B_4 = (\mathsf{E}x)(\mathsf{E}y)[R(x,y) \ \wedge \ R(y,x)]. \tag{12}$$

[1] In this problem (A1) we use '(x)' as the object language *universal* quantifier over variable 'x' and '$(\mathsf{E}x)$' as the object language *existential* quantifier over variable 'x'.

A2 **Logic** (25 points)

Translate each of these sentences into a suitable first order predicate logic language *with equality*. Of course you should *first specify that language* and *define an interpretation of its symbols* of relevance to your translation.

Your specified language should include *among other things* symbols of the appropriate types (you must say *which* types) for expressing 'is a professor', 'is happy', 'is a student of', 'likes logic', and 'Professor Smith'.

a. (5 points)
   Each professor is happy if some of his/her students like logic.

b. (5 points)
   Some professor is happy if all of his/her students like logic.

c. (5 points)
   Some professor is happy if some of his/her students like logic.

d. (5 points)
   Professor Smith is happy if some of her students like logic.

e. (5 points)
   The professor is happy if all her students like logic.

A3 **Logic** (25 points)

You need not do both A3 part a and A3 part b to get full credit for just one of them. Of course you need to do both to get credit for both. (☺)

a. (12.5 points)

Prove *in detail* that, for any axiom-scheme/rule-based system for a propositional calculus[2] which has *Modus Ponens* as its *only* rule of inference[3] and which has *at least* the three axiom schemes

$$(\alpha \supset (\beta \supset \alpha)), \tag{13}$$

$$((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))), \tag{14}$$

and

$$(\alpha \supset \alpha),^{4} \tag{15}$$

that we also must have (for this system) the following

Deduction Theorem $[(\Gamma \cup \{\alpha\}) \vdash \beta] \Rightarrow [\Gamma \vdash (\alpha \supset \beta)]$.

Hint: Proceed by mathematical induction on n = the number of steps in a formal proof, $\beta_1, \ldots, \beta_n$, of β from $(\Gamma \cup \{\alpha\})$. (Therefore, $\beta = \beta_n$.)
Show how to convert this formal proof to a formal proof of $(\alpha \supset \beta)$ from Γ. In this interest consider the three cases as follows.
Case (1). β_n is an axiom or a member of Γ.
Case (2). $\beta_n = \alpha$.
Case (3). β_n is obtained from *prior* steps in $\beta_1, \ldots, \beta_n$ by Modus Ponens. (This is the case requiring the induction hypothesis.)

b. (12.5 points)

Explicitly use the Deduction Theorem of A3 part a to show that, in the system of A3 part a,

$$\vdash ((\alpha \supset (\beta \supset \gamma)) \supset (\beta \supset (\alpha \supset \gamma))). \tag{16}$$

No credit if you do not use the Deduction Theorem.

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[2]Such systems are sometimes called *Hilbert-style*.

[3]*Modus Ponens* is (by definition) the rule for inferring β (from any set of premises) from already having inferred (from that same set of premises) both α and $\alpha \supset \beta$.

[4]Some of you may know that the third axiom scheme is, in this system, derivable from the other two — employing Modus Ponens. Do *not* prove or use this fact to solve this problem (A3).

A4 **Logic** (25 points)

Here is one form of the *Compactness Theorem* for (first order) predicate logics.

Theorem (Compactness) Suppose ℓ is a first order predicate logic language. Suppose Γ is a set of formulas of ℓ.

Then: Γ has a model $\Leftrightarrow (\forall$ *finite* $\Delta \subseteq \Gamma)[\Delta$ has a model$]$.

In *this* problem, you may and should *explicitly employ* this form of the Compactness Theorem with*out* proof.

Definition

1. $(U, <)$ is a *partial ordering on U* $\overset{\text{def}}{\Leftrightarrow}$ U is a set and $<$ is a binary relation on U such that

$$(\forall u, v \in U)[u < v \Rightarrow v \not< u] \tag{17}$$

   and

$$(\forall u, v, w \in U)[[u < v \text{ and } v < w] \Rightarrow u < w]. \tag{18}$$

2. $(U, <)$ is a *total ordering on U* $\overset{\text{def}}{\Leftrightarrow}$ $(U, <)$ is a partial ordering on U and

$$(\forall u, v \in U \mid u \neq v)[u < v \text{ or } v < u]. \tag{19}$$

Examples

1. Suppose U is the set of nodes, $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in the finite graph depicted in Figure 1 below and $<$ on U is the transitive closure of the (directed) edge relation depicted by arrows in this figure. For *example*, in Figure 1, we have $5 < 2$.
   This $(U, <)$ *is* a total ordering on U.

```
5 --> 3 --> 6 --> 8 --> 7 --> 4 --> 1 --> 2
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Figure 1: Example 1

2. Suppose U is the set of nodes, $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in the finite graph depicted in Figure 2 below and $<$ on U is the transitive closure of the (directed) edge relation depicted by arrows in this figure. For *example*, in Figure 2, we have $5 < 1$. This $(U, <)$ is partial ordering on U but *not* a total ordering on U. It *does* satisfy (17) and (18) above; however, it does *not* satisfy (19). For *example*, in Figure 2, we have $1 \not< 2$ and $2 \not> 1$, and, also, $4 \not< 7$ and $7 \not> 4$.

```
          8 --> 2

    5 --> 3 --> 4
          |
           ---> 7
                |
                V
          6 --> 1
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Figure 2: Example 2

(Problem A4 continues onto the next page.)

(This page is the continuation of Problem A4.)

3. Suppose U is the set of all real numbers and $<$ on U stands for the ordinary less-than relation on U. This $(U,<)$ *is* a total ordering on U.
4. Suppose U is the *set of all subsets of* the set of real numbers. Then $(U,\subset)$ is a partial ordering on U. You may find it hard to think of an example $<$ so that $(U,<)$ is a *total order* on U.

Definition Suppose $(U,<)$ and $(U,<')$ are each partial orderings on U.

$(U,<')$ *extends* $(U,<)$ $\overset{\text{def}}{\Leftrightarrow}$

$$(\forall u,v \in U)[u < v \Rightarrow u <' v]. \tag{20}$$

E.g., Example 1 above extends Example 2 above.

Explicitly apply the form of the Compactness Theorem above and employ the Hint below to prove

$$(\forall \text{ partial orderings } (U,<))(\exists <')[(U,<') \text{ is a total ordering on } U \text{ which extends } (U,<)]. \tag{21}$$

Hint: Suppose $(U,<)$ is a partial ordering on U.

Define a first order predicate logic language ℓ and a set of formulas Γ of ℓ such that

(a) ℓ contains a constant symbol c_u for each $u \in U$ and a binary predicate symbol $<$; and

(b) Γ contains *two* axioms expressing (17) and (18) above, respectively, *and all the axioms of the two forms*

$$c_u < c_v, \tag{22}$$

where $u,v \in U$ *and* $u < v$, *and*

$$[c_u < c_v \ \vee \ c_v < c_u], \tag{23}$$

where $u,v \in U$ *and* $u \neq v$.

Suppose V is any *finite* subset of U. Show that, for the partial ordering (V, $<$ restricted to V), there is a total ordering $(V,<')$ extending it. To do this, suppose $V = \{v_1,\ldots,v_n\}$, $n \geq 0$. Pick out the $<$-minimal elements of V and order them by their v-subscripts. Pull them out of V and iterate until no longer possible. String the results of each iteration after one another to (graphically) construct $<'$.[5] *Make this construction more detailed and convincing.*

Suppose Δ is any *finite* subset of Γ. Consider *which* constant symbols c_u are explicitly mentioned in this Δ. Obtain a model for Δ.

Next apply the form of the Compactness Theorem above to obtain a model of Γ.

Consider:

- the interpretation *in this latter model* of each constant symbol c_u, for $u \in U$, *and*
- the interpretation *in this latter model* of the binary predicate symbol $<$.

[5] For Example 2 above the $<$-minimal elements of *that* U are $8, 5, 6$. Following the iterative procedure on *this* example will yield a total ordering on this U extending $(U,<)$ different from the one in Example 1.