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Reality and Differential Calculus on the q-Euclidean Space*

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Abstract By introducing the left and right derivatives, we establish a real structure for the covariant differential calculus on the N-dimensional quantum Euclidean space. The method is also applicable to the quantum Minkowski space.

Keywords: q-Euclidean space, reality and differential calculus.

1 Introduction

Recently, the study on the covariant differential calculus on the quantum hyperplane developed by Wess and Zumino[1] has made much headway. In Refs. [2,3], Wess-Zumino's method is generalized to some more universal cases where the transformation groups acting on the quantum planes include, in addition to $GL_q(N)$, both $SO_q(N)$ and $SP_q(2n)$. Therefore, using these differential calculi, the possible q-deformation of some physical systems, e.g. 3-dimensional oscillator[2] and hydrogen-like atoms[4] can be discussed. Moreover, people are trying to relate this non-commutative space to the microscopic structure of our space-time. As is well known, the physical space-time requires some reality conditions compatible with the differential calculus on it. Up to now, however, such physically required reality conditions have not been well established yet with the differential calculus on the quantum (pseudo) Euclidean space. In this paper, we present a kind of real structure, which is different from the one given by Wess and Zumino for the GL_q case[1] and for the differential calculus on the quantum Euclidean space within the framework of C^*-algebra. As pointed out by Woronowicz, the matrix elements of the quantum group belong to a C^*-subalgebra[7]. This structure also exists in the algebra consisting of the coordinates of the quantum plane, the carried space of the quantum group[8]. The operator $*$, which is an antihomomorphism in a C^*-algebra, i.e. $(ab)^* = b^*a^*$, can be considered as something like the complex conjugation in quantum mechanics. Therefore, by distinguishing the action of derivatives of the left from that of the right, one can give a suitable definition of the complex conjugation under which the entire scheme goes into itself.

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2 Differential Calculus on the Quantum Plane

The quantum plane is defined, according to Manin[8], in terms of N variables (coordinates) x^i, $i = 1, 2, \cdots, N$, which belong to a noncommutative associative C^*-algebra and satisfy the commutation relations

$$x^i x^j - B^{ij}_{kl} x^k x^l = 0, \tag{1}$$

where B^{ij}_{kl} are numerical coefficients in matrix form. The (left) derivatives are defined as usual

$$\partial_i \equiv \frac{\partial}{\partial x^i}, \quad (\partial_i x^j) = \delta^j_i. \tag{2}$$

The commutation relations between the coordinates and the derivatives and the commutation relations among the derivatives are assumed to be

$$\partial_i x^j = \delta^j_i + C^{jk}_{il} x^l \partial_k, \tag{3}$$

$$\partial_i \partial_j = \partial_l \partial_k F^{kl}_{ji}, \tag{4}$$

respectively, where C and F, like B, are also numerical matrices. The consistency conditions of the theory are summarized as follows[1]:

$$\begin{gathered}(E_{12} - B_{12})(E_{12} + C_{12}) = 0, \quad B = F \\ B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23}, \\ B_{23}C_{12}C_{23} = C_{12}C_{23}B_{12}.\end{gathered} \tag{5}$$

The last two formulae of (5) are like Yang-Baxter equations indicating that B, C and F must be some combinations of the $\check{R}$ matrix which is the solution of the braid Yang-Baxter equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}. \tag{6}$$

For $SO_q(N)$ and $SP_q(2n)$ solutions to the Yang-Baxter equation, the $\check{R}$ matrix has three different eigenvalues[9].

$$(\check{R} - \lambda_2)(\check{R} - \lambda_1)(\check{R} - \lambda_0) = 0. \tag{7}$$

For example,

$$\lambda_2 = q, \quad \lambda_1 = -q^{-1}, \quad \lambda_0 = q^{-2} \tag{8}$$

corresponding to the quintet, the triplet and the singlet respectively in B_1 case. The projection operators of the three eigenvalues are constructed as follows:

$$\begin{aligned} Q^{(2)} &= \frac{(\check{R} - \lambda_0)(\check{R} - \lambda_1)}{(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)}, \\ Q^{(1)} &= \frac{(\check{R} - \lambda_0)(\check{R} - \lambda_2)}{(\lambda_1 - \lambda_0)(\lambda_1 - \lambda_2)}, \\ Q^{(0)} &= \frac{(\check{R} - \lambda_1)(\check{R} - \lambda_2)}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)}, \end{aligned} \tag{9}$$

with the properties

$$Q^{(\alpha)} Q^{(\beta)} = \delta^{\alpha\beta} Q^{(\beta)}, \quad Q^{(0)} + Q^{(1)} + Q^{(2)} = E \tag{10}$$

and

$$\check{R}=\lambda_2 Q^{(2)}+\lambda_1 Q^{(1)}+\lambda_0 Q^{(0)},\ \check{R}^{-1}=\lambda_2^{-1} Q^{(2)}+\lambda_1^{-1} Q^{(1)}+\lambda_0^{-1} Q^{(0)}. \qquad (11)$$

It is easy to see from the Yang-Baxter equation (6) and relations (9) that

$$Q_{12}^{(a)}\check{R}_{23}\check{R}_{12}=\check{R}_{23}\check{R}_{12}Q_{23}^{(a)},\ \check{R}_{12}\check{R}_{23}Q_{12}^{(a)}=Q_{23}^{(a)}\check{R}_{12}\check{R}_{23}. \qquad (12)$$

Then it is straightforward to show that conditions in (5) are satisfied if one takes[3,4]

$$C=-\lambda_1^{-1}\check{R}=q\check{R},\ \text{or}\ -\lambda_1\check{R}^{-1}=q^{-1}\check{R}^{-1}, \qquad (13)$$

(we choose the second one in this paper for definiteness) and

$$B=F=E-Q^{(1)}. \qquad (14)$$

As stated in Refs. [2, 4, 10], there exist a metric g and its inverse for the quantum Euclidean space, which are the left-acting and right-acting singlet eigenvectors of $\check{R}$ matrix, respectively

$$\check{R}^{ij}_{kl}g^{kl}=\lambda_0 g^{ij},\ g_{ij}\check{R}^{ij}_{kl}=\lambda_0 g_{kl} \qquad (15)$$

and satisfy the following useful relations[2,10]

$$g_{ij}g^{jk}=\delta_i^k,\ g^{ij}g_{jk}=\delta_k^i, \qquad (16)$$

$$g_{ij}\check{R}^{jk}_{lm}=\check{R}^{-1kn}_{il}g_{nm},\ \check{R}^{ij}_{kl}g^{lm}=g^{in}\check{R}^{-1jm}_{nk}. \qquad (17)$$

With the help of g, we can raise the indices of the derivatives as

$$\partial^i=g^{ij}\partial_j \qquad (18)$$

and write relations (3) and (4) in the following form

$$\partial^i x^j=g^{ij}+q^{-1}\check{R}^{ij}_{kl}x^k\partial^l, \qquad (3')$$

$$\partial^i\partial^j=F^{ij}_{kl}\partial^k\partial^l, \qquad (4')$$

where (13) and (14) are utilized.

3 Algebra With Right Derivatives

We now introduce the right derivatives $\tilde{\partial}$ as follows:

$$(x^i{}_j\tilde{\partial})=\delta^i_j. \qquad (19)$$

Their commutation relations with the coordinates and among themselves are

$$x^i{}_j\tilde{\partial}=\delta^i_j+{}_k\tilde{\partial}x^l\tilde{C}^{ki}_{lj}, \qquad (20)$$

$${}_i\tilde{\partial}_j\tilde{\partial}={}_k\tilde{\partial}_l\tilde{\partial}\tilde{F}^{lk}_{ji}, \qquad (21)$$

where $\tilde{C}$ and $\tilde{F}$ are numerical matrices to be determined later.

Letting $\tilde{\partial}$ act on the basic relation (1) from the right, one can get the consistency condition by the two sides to be equal

$$(E+\tilde{C})(E-B)=0,\ B_{23}\tilde{C}_{12}\tilde{C}_{23}=\tilde{C}_{12}\tilde{C}_{23}B_{12}. \qquad (22)$$

In the same way for $x\tilde{\partial}\tilde{\partial}$, one can get

$$(E-\tilde{F})(E+C)=0,\ \tilde{F}_{12}C_{23}C_{12}=C_{23}C_{12}\tilde{F}_{23}. \tag{23}$$

These conditions mean that $C=\tilde{C}$ and $F=\tilde{F}$. From now on, we will not distinguish them any more.

Similar to the relations in (3′) and (4′), we have the alternative form of (20) and (21)

$$x^i\tilde{\partial}^j = g^{ij} + q^{-1}\tilde{\partial}^k x^l \check{R}^{ij}_{kl}, \tag{20′}$$

$$\tilde{\partial}^i\tilde{\partial}^j = \tilde{\partial}^k\tilde{\partial}^l F^{ij}_{kl} \tag{21′}$$

by raising the indices of $\tilde{\partial}$ and denoting them by

$$\tilde{\partial}^i = {}_j\tilde{\partial}g^{ji}. \tag{24}$$

To complete the enlarged algebra, we define the commutation relations between the right derivatives and the left ones as

$$\partial^i\tilde{\partial}^j = G^{ij}_{kl}\tilde{\partial}^k\partial^l. \tag{25}$$

Multiplying this equation from the right by x and commuting x through to the left we find terms linear in the derivatives which must be cancelled out separately. This requires

$$q^2C^{ij}_{kl}g^{lm} = g^{il}G^{jm}_{lk}. \tag{26}$$

It is easy to see that the remaining terms will also be cancelled out if

$$C_{12}C_{23}^{-1}G_{12} = G_{23}C_{12}^{-1}C_{23}. \tag{27}$$

Due to the fact that $C=-\lambda_1\check{R}^{-1}$, we can easily see that $G=C^{-1}=q\check{R}$, by comparing (26) with (17). Consequently (27) is satisfied.

One can still perform various consistency checks and find out that all those conditions are nothing more than the Yang-Baxter-like equations. These equations will guarantee the associativity of any triple product of the algebraic objects. Eqs. (1),(3′),(4′),(20′),(21′) and (25) complete our enlarged algebra.

4 Complex Conjugation and Transformation Properties

Let us now consider the real structure of the quantum Euclidean space. In the following discussion we require q to be real, i.e. $q^*=q$. Bearing in mind that x^i are q-deformed "spherical harmonical coordinates", we define the complex conjugation as in the classical case

$$(x^i)^* = x^jg_{ji}. \tag{28}$$

The complex conjugation of the derivatives is accordingly defined as

$$(\partial^i)^* = \tilde{\partial}^jg_{ji},\ (\tilde{\partial}^i)^* = \partial^jg_{ji}, \tag{29}$$

with the exchange of the acting directions considered.

It is easy to check that the complex conjugation described above is an involution, e.g.

$$((x^i)^*)^* = (x^jg_{ji})^* = (g_{ji})^*(x^j)^* = g^{ji}x^kg_{kj} = x^k\delta^i_k = x^i, \tag{30}$$

where we have the convention

$$(g^{ij})^* = g_{ij}, \quad (g_{ij})^* = g^{ij}. \tag{31}$$

One can easily see that under the complex conjugations (3')⟷(20'), (4')⟷(21'), while (1) and (25) go into themselves respectively. This means that the whole enlarged algebra is generated by coordinates, and the left and right derivatives are compatible with the * operator.

To end this section, we would like to consider the transformation behaviour of these algebraic objects under the $SO_q(N)$ action. The transformation matrix M of the quantum group satisfies the well-known Yang-Baxter relation[9]

$$\check{R}_{12}M_1M_2 = M_1M_2\check{R}_{12}. \tag{32}$$

For the orthogonal group, M still satisfies[9,10]

$$g_{ij}M^i_kM^j_l = g_{kl}, \quad g^{kl}M^i_kM^j_l = g^{ij}. \tag{33}$$

One can see immediately that all the relations of the differential calculus on the quantum Euclidean space developed above are covariant under the action of $SO_q(N)$ transformation

$$x^i \to M^i_jx^j, \quad \partial^i \to M^i_j\partial^j, \quad \tilde{\partial}^i \to M^i_j(\tilde{\partial}^j). \tag{34}$$

The two kinds of derivatives transform differently in their lower index form

$$\partial_i \to (M^t)^{-1j}_i\partial_j, \quad {}_i\tilde{\partial} = {}_j\tilde{\partial}M^{-1j}_i, \tag{35}$$

for $(M^t)^{-1} \neq (M^{-1})^t$ when $q \neq 1$. In this way, we have

$$\delta^j_i = \partial_ix^j \to (M^t)^{-1k}_i(\partial_kx^l)(M^t)^j_l = \delta^j_i, \tag{36a}$$

and

$$\delta^j_i = x^j{}_i\tilde{\partial} \to M^j_k(x^k{}_l\tilde{\partial})M^{-1l}_i = \delta^j_i. \tag{36b}$$

Therefore, taking the transformation properties into account, a consistent calculus should include two kinds of the derivatives as required by covariance. For the real quantum Euclidean space, the matrix elements M^i_j must be real in the sense that

$$(M^i_j)^* \equiv \overline{M}^j_i = g^{jk}M^l_kg_{li}. \tag{37}$$

Equipped with this kind of real structure, the q-Euclidean space will be more justified, and consequently the q-Schrödinger equation with the differential calculus presented above, along with its solutions such as the q-Oscillator[2] and the q-hydrogen-like atoms[4] will be more closely related to its classical counterpart. It is easy to see that the entire scheme goes back to the ordinary calculus in the limit $q \to 1$. The discussion given here on the real structure of the quantum Euclidean space is applicable to the quantum Minkowski space and the q-Dirac equation.

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