

QUANTUM LIE SUPERALGEBRAS AND “NON-STANDARD” BRAID GROUP REPRESENTATIONS*

LI LIAO^{1,2} and XING-CHANG SONG^{1,2,3}

¹CCAST (World Laboratory), P. O. Box 8730, Beijing 100080, China

²Department of Physics, Peking University, Beijing 100871, China

³Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China

Received 29 September 1990

Revised 16 January 1991

The formulae of the Faddeev–Reshetikhin–Takhtajan (FRT) method in supersymmetric case are presented transparently and consistently. With the help of these formulae, the simplest “non-standard” solution of braid group representation (BGR) is re-examined. The result shows that the hidden symmetry associated with this “non-standard” BGR is indeed the q -deformed Lie superalgebra $U_q(\mathfrak{gl}(1|1))$.

1. Introduction

Recently, much attention has been paid to the so-called quantum groups, or the quantum universal enveloping algebras,¹ which were originated from the research of trigonometric/hyperbolic solutions of the quantum Yang–Baxter equations (YBE).² And it has been shown that they are closely related to various physically interesting models and theories, such as the exactly soluble statistical models,² inverse scattering method for nonlinear evolution equation,³ factorizable s matrix and integrable field theory,⁴ conformal field theory and topological Chern–Simons theory.⁵

The mathematical structure of the quantum groups has been systematically carried out by Drinfeld,¹ Jimbo¹ and Reshetikhin *et al.*⁶ More recently, efforts have been made in generalizing this structure to include supersymmetric case.⁷

By now it is well-known that the braid group representations (BGR) are usually obtained from the trigonometric/hyperbolic solutions of YBE by setting the spectral parameter to infinity $\theta \rightarrow \pm\infty$. The standard method in obtaining BGR from the universal R -matrix⁸ for the standard q -deformation of usual Lie algebras gives a series of BGRs referred to as “standard” ones, which approach to the permutation matrix when the deformation parameter $q \rightarrow 1$, and whose first order terms satisfy the classical YBE. The simplest example appears in the six vertex model

* This work is supported in part by the National Natural Science Foundation of China, Institute of High Energy Physics and Institute of Theoretical Physics, Academia Sinica.

† Permanent address.

$$\check{R}_{12} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \xrightarrow{q \rightarrow 1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P_{12}, \quad (1.1)$$

where $d = q - q^{-1}$. However, a lot of “non-standard” BGRs have also been found recently.⁹ These representations have, among other different properties, the limit behavior different from that of the “standard” ones. For the simplest two-state model, the “non-standard” fundamental R -matrix takes the form

$$\check{R}_{12}^{(n)} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \xrightarrow{q \rightarrow 1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathcal{P}_{12} = P_{12} \eta_{12} \quad (1.2)$$

with η_{12} a phase factor being diagonal $(1, 1, 1, -1)$. It is not difficult to recognize that \mathcal{P}_{12} in Eq. (1.2) is not the usual permutation matrix but the super (or graded) permutation matrix acting upon a superspace whose first basis is bosonic and the second one fermionic. This implies that the “non-standard” $\check{R}_1^{(n)}$ may be closely related to a Lie superalgebra. We show in this paper that it is indeed the case. Our result is different from that reported in a recent paper.¹⁰ By employing the usual Faddeev–Reshetikhin–Takhtajan (FRT) constructive method, these authors claim that they obtain a peculiar new quantum group whose classical limit is not a Lie superalgebra despite that some of its relations appear as fermion-like. The essential point of our analysis is, to deal with a would-be super BGR one must start from formulae appropriate for the super case at the very beginning, and must take care of the consistency throughout the whole procedure of analysis.

So we begin with the supersymmetric formulae for FRT’s method in Sec. 2 and then turn to discuss the concrete example Eq. (1.2) in Sec. 3. In Sec. 4 brief results for other “non-standard” BGRs are reported and a short discussion is given.

2. FRT’s Method in Supersymmetric Case

According to Ref. 8, algebra $A(R)$ is defined in terms of unity 1 and generators t_i^j constituting a matrix T formally associated with a linear space V and satisfying the commutation relations written most naturally in the matrix form

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}, \quad (2.1)$$

where R is a numerical matrix associated with $V \otimes V$, supposed to be non-singular, whereas T_1 and T_2 are operator-valued matrices

$$T_1 = T \otimes I, \quad T_2 = I \otimes T = P_{12} T_1 P_{12}, \quad (2.2)$$

where I is the unit matrix in V and P_{12} is the permutation matrix in $V \otimes V$ as mentioned in Sec. 1. Then the Yang–Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.3}$$

can be considered (sufficiently) as the compatibility condition for the main relation (2.1). This comes from the associativity of the triple product $T_1T_2T_3$ in $V^{\otimes 3}$ in different order, where

$$\begin{aligned} T_1 &= T \otimes I \otimes I, T_2 = I \otimes T \otimes I = P_{12}T_1P_{12}, \\ T_3 &= I \otimes I \otimes T = P_{23}T_2P_{23}. \end{aligned} \tag{2.4}$$

The dual algebra $B(R)$ is generated by 1 and a set of generators $l^{(\pm)}_i$ comprising two matrices $L^{(\pm)}$ and satisfying the relations

$$R_{21}L_1^{(\varepsilon)}L_2^{(\varepsilon')} = L_2^{(\varepsilon')}L_1^{(\varepsilon)}R_{21}, \tag{2.5}$$

with $(\varepsilon, \varepsilon') = (+, +), (+, -)$ or $(-, -)$. Further restriction is introduced by the duality condition

$$\langle L_1^{(\pm)}, T_2 \rangle = R_{12}^{(\pm)}, \tag{2.6}$$

where \langle, \rangle means pairing between $A(R)$ and $B(R)$, and

$$R_{12}^{(+)} \equiv R_{21} = P_{12}R_{12}P_{12}, R_{12}^{(-)} = R_{12}^{-1}. \tag{2.7}$$

As expected, the condition of pairing effectively reduces the number of non-zero generators $l^{(\pm)}_i$ so that the total number of generators of $A(R)$ and $B(R)$ is equal. Usually $L^{(+)}$ and $L^{(-)}$ are respectively taken to be upper- and lower-triangular matrices with operator entries. The Yang–Baxter relation (2.3) shows that the pairing is consistent with the main commutation relations (2.1) and (2.5). It is well-known that by introducing a corresponding braid matrix \check{R} instead of R in the following way

$$R_{12} = \check{R}_{12} P_{12}, \tag{2.8}$$

one can put the Yang–Baxter relation (2.3) into its braid form

$$\check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23}, \tag{2.9}$$

whereas relation (2.5) become

$$\check{R}_{12} L_1^{(\varepsilon)} L_2^{(\varepsilon')} = L_1^{(\varepsilon')} L_2^{(\varepsilon)} \check{R}_{12}. \tag{2.10}$$

Now we turn to discuss the supersymmetric case. In this case, space V is graded. Therefore care must be taken in dealing with tensor product. say

$$(A \otimes B)(C \otimes D) = (-)^{\text{deg}B \cdot \text{deg}C} AC \otimes BD.$$

Taking into account this property, all the relations given above remain to be true. Attention must be paid to these relations which are not explicitly in the tensor form, e.g., YBE(2.3) or (2.9). Here we give the explicit supersymmetric form. The main relation (2.1) takes the form

$$\mathcal{R}_{12} \mathcal{T}_1 \mathcal{T}'_2 = \mathcal{T}'_2 \mathcal{T}_1 \mathcal{R}_{12} \tag{2.11}$$

with

$$\begin{aligned} \mathcal{T}_1 &= \mathcal{T} \otimes I, \\ \mathcal{T}'_2 &= \mathcal{P}_{12} \mathcal{T}_1 \mathcal{P}_{12} = \eta_{12} (I \otimes \mathcal{T}) \eta_{12} = \eta_{12} \mathcal{T}_2 \eta_{12}, \end{aligned} \tag{2.12}$$

where $\mathcal{P}_{12} = P_{12} \eta_{12}$, $(\eta_{12})_{i_1 i_2}^{j_1 j_2} = (-1)^{\text{deg}(i_1) \text{deg}(i_2)} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2}$. And the YBE now becomes

$$\eta_{12} \mathcal{R}_{12} \eta_{13} \mathcal{R}_{13} \eta_{23} \mathcal{R}_{23} = \eta_{23} \mathcal{R}_{23} \eta_{13} \mathcal{R}_{13} \eta_{12} \mathcal{R}_{12} \tag{2.13}$$

which can be obtained by considering the associativity of the triple product $\mathcal{T}_1 \mathcal{T}'_2 \mathcal{T}''_3$ in $V^{\otimes 3}$, where

$$\begin{aligned} \mathcal{T}_1 &= \mathcal{T} \otimes I \otimes I, \\ \mathcal{T}'_2 &= \mathcal{P}_{12} \mathcal{T}_1 \mathcal{P}_{12} = \eta_{12} (I \otimes \mathcal{T} \otimes I) \eta_{12} = \eta_{12} \mathcal{T}_2 \eta_{12}, \\ \mathcal{T}''_3 &= \mathcal{P}_{23} \mathcal{T}'_2 \mathcal{P}_{23} = \eta_{23} \eta_{13} (I \otimes I \otimes \mathcal{T}) \eta_{13} \eta_{23} = \eta_{23} \eta_{13} \mathcal{T}_3 \eta_{13} \eta_{23}. \end{aligned} \tag{2.14}$$

The dual algebra $B(R)$ is characterized by the relations

$$\mathcal{R}_{21} \mathcal{L}_1^{(\varepsilon)} (\eta_{12} \mathcal{L}_2^{(\varepsilon')} \eta_{12}) = (\eta_{12} \mathcal{L}_2^{(\varepsilon')} \eta_{12}) \mathcal{L}_1^{(\varepsilon)} \mathcal{R}_{21} \tag{2.15}$$

with $(\varepsilon, \varepsilon') = (+, +), (+, -)$ or $(-, -)$. And the pairing between $A(R)$ and $B(R)$ now takes the form

$$\langle \mathcal{L}_1^{(\pm)}, \mathcal{T}_2 \rangle = \mathcal{R}_{12}^{(\pm)}, \tag{2.16}$$

where

$$\mathcal{R}_{12}^{(+)} = \mathcal{P}_{12} \mathcal{R}_{12} \mathcal{P}_{12} = \eta_{12} \mathcal{R}_{21} \eta_{12}, \quad \mathcal{R}_{12}^{(-)} = \mathcal{R}_{12}^{-1}. \tag{2.17}$$

Still the Yang–Baxter relation (2.13) ensures the consistency between the pairing (2.16) and the main commutation relations (2.11) and (2.15). Now if we introduce the braid matrix $\check{\mathcal{R}}$ by setting

$$\mathcal{R}_{12} = \check{\mathcal{R}}_{12} \mathcal{P}_{12} = \check{\mathcal{R}}_{12} P_{12} \eta_{12}, \tag{2.18}$$

we can put the Yang–Baxter relation (2.13) in the form

$$\check{\mathcal{R}}_{12} \check{\mathcal{R}}_{23} \check{\mathcal{R}}_{12} = \check{\mathcal{R}}_{23} \check{\mathcal{R}}_{12} \check{\mathcal{R}}_{23}, \tag{2.19}$$

which is just Eq. (2.9) with $\mathcal{R} = R$. Meanwhile relations (2.15) now become

$$\check{R}_{12} (\eta_{12} \mathcal{L}_1^{(\varepsilon)} \eta_{12}) \mathcal{L}_2^{(\varepsilon')} = (\eta_{12} \mathcal{L}_1^{(\varepsilon')} \eta_{12}) \mathcal{L}_2^{(\varepsilon)} \check{R}_{12} \tag{2.20}$$

with $(\varepsilon, \varepsilon') = (+, +), (+, -)$ or $(-, -)$.

The derivation of Eqs. (2.11)–(2.20) is completely parallel to that of Eqs. (2.1)–(2.10). We have made use of several important relations which came from the diagonal property of phase factor η_{ab} ,

$$\eta_{ab} = \eta_{ba}, \quad \eta_{ab} \eta_{cd} = \eta_{cd} \eta_{ab}, \tag{2.21}$$

and from the “weight conservation” condition on \mathcal{R} -matrix

$$\eta_{ac} \eta_{bc} \mathcal{R}_{ab} = \mathcal{R}_{ab} \eta_{ac} \eta_{bc} . \tag{2.22}$$

Note that, in Eqs. (2.11)–(2.20) the grading property has been taken into account by introducing the factor η_{ab} , and all matrices can be considered as the ordinary ones. But for relations other than these equations one must take care of the grading property in dealing with the multiplication of graded quantities.

The coincidence of Eq. (2.19) with Eq. (2.9) implies that both the ordinary R -matrices satisfying YBE (2.3) and the supersymmetric R -matrices satisfying super YBE (2.13) give the same set of braid group relation. So it is not surprising that some sets of braid group representations may connect with the q -deformation of ordinary Lie algebras and other sets of braid group representations with the q -deformation of Lie superalgebras.

3. Quantum Superalgebra $U_q(\mathfrak{gl}(1|1))$ Corresponding to the Simplest “Non-Standard” BGR

As mentioned in Sec. 1, the simplest “non-standard” BGR (1.2) has the limit behavior (as $q \rightarrow 1$) as the graded permutation matrix. So it may concern a supersymmetry. If it does, one must use the formulae presented in the second half of Sec. 2 to analyze it. Consider Eq. (2.20) with \check{R} given by (1.2), i.e.,

$$\begin{aligned} \check{R}_{11}^{11} &= q, \check{R}_{22}^{22} = -q^{-1}, \check{R}_{12}^{21} = \check{R}_{21}^{12} = 1, \\ \check{R}_{21}^{21} &= q - q^{-1}, \check{R}_{ik}^{jl} = 0, \text{ otherwise .} \end{aligned} \tag{3.1}$$

$L^{(\pm)}$ are respectively taken to be upper- and lower-triangular 2×2 matrices with operator entries

$$L_2^{(+1)} = 0, L_1^{(-2)} = 0 . \tag{3.2}$$

By straightforward calculation it can easily be seen that Eq. (2.20) lead to the following relations

$$[L^{(\varepsilon)a}, L^{(\varepsilon')b}] = 0, \varepsilon, \varepsilon' = \pm , \tag{3.3}$$

$$L^{(+1)}_1 L^{(+2)}_1 = q^{-1} L^{(+2)}_1 L^{(+1)}_1, \tag{3.4a}$$

$$L^{(+2)}_2 L^{(+1)}_1 = q^{-1} L^{(+1)}_1 L^{(+2)}_2, \tag{3.4b}$$

$$L^{(-1)}_1 L^{(-1)}_2 = q^{-1} L^{(-1)}_2 L^{(-1)}_1, \tag{3.5a}$$

$$L^{(-2)}_2 L^{(-1)}_2 = q^{-1} L^{(-1)}_2 L^{(-2)}_2, \tag{3.5b}$$

$$(L^{(+2)}_1)^2 = 0, \tag{3.6a}$$

$$(L^{(+1)}_1)^2 = 0, \tag{3.6b}$$

$$L^{(+1)}_1 L^{(-1)}_2 = q L^{(-1)}_2 L^{(+1)}_1 . \tag{3.7a}$$

$$L^{(+2)}_2 L^{(-1)}_2 = q L^{(-1)}_2 L^{(+2)}_2, \tag{3.7b}$$

$$L^{(-1)}_1 L^{(+2)}_1 = q L^{(+2)}_1 L^{(-1)}_1, \tag{3.8a}$$

$$L^{(-2)}_2 L^{(+1)}_1 = q L^{(+1)}_1 L^{(-2)}_2, \tag{3.8b}$$

$$\{L^{(-\frac{1}{2}} L^{(+2)}_1\} = (q - q^{-1})(L^{(-2)}_2 L^{(+1)}_1 - L^{(+2)}_2 L^{(-1)}_1). \tag{3.9}$$

It is easy to verify that $L^{(+1)}_1 L^{(-1)}_1 = L^{(-1)}_1 L^{(+1)}_1$ and $L^{(+2)}_2 L^{(-2)}_2 = L^{(-2)}_2 L^{(+2)}_2$ belong to the center of the algebra. So they can be chosen as unit 1. Writing

$$L^{(+)} = \begin{pmatrix} k^{-1} & (q - q^{-1})x \\ 0 & l \end{pmatrix}, L^{(-)} = \begin{pmatrix} k & 0 \\ (q - q^{-1})y & l^{-1} \end{pmatrix} \tag{3.10}$$

then we have the following relations

$$[k, l] = 0, \tag{3.11}$$

$$kxk^{-1} = qx, lxl^{-1} = q^{-1}x, \tag{3.12}$$

$$kyk^{-1} = q^{-1}y, lyl^{-1} = qy, \tag{3.13}$$

$$x^2 = y^2 = 0, \tag{3.14}$$

$$\{x, y\} = \frac{1}{q - q^{-1}}(kl - k^{-1}l^{-1}). \tag{3.15}$$

Setting

$$k = q^N, l = q^M \tag{3.16}$$

then we have the alternative form of the (anti)-commutation relations

$$[N, x] = x, [M, x] = -x \tag{3.12'}$$

$$[N, y] = -y, [M, y] = y \tag{3.13'}$$

and

$$\{x, y\} = [N + M]_q = \frac{q^{N+M} - q^{-N-M}}{q - q^{-1}}. \tag{3.15'}$$

The algebra generated by 1, k^\pm , l^\pm , x and y given in Eqs. (3.11)–(3.15) is nothing but the quantum universal enveloping algebra of super Lie algebra $gl(1|1)$, which reduces to $gl(1|1)$ in the limit $q \rightarrow 1$. The co-product of this algebra can be taken as

$$\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}, \Delta(l^{\pm 1}) = l^{\pm 1} \otimes l^{\pm 1}, \tag{3.17}$$

$$\Delta(x) = x \otimes k + l^{-1} \otimes x, \Delta(y) = y \otimes l + k^{-1} \otimes y. \tag{3.18}$$

And the co-unit ε and anti-pode S are defined as

$$\varepsilon(x) = \varepsilon(y) = 0, \varepsilon(k) = \varepsilon(l) = 1. \tag{3.19}$$

$$S(k) = k^{-1}, S(l) = l^{-1}, S(x) = -l x k^{-1}, S(y) = -k y l^{-1}. \tag{3.20}$$

One can easily verify that Δ and ϵ are algebra homomorphism and S an anti-morphism. Therefore $U_q(\mathfrak{gl}(1|1))$ is a Hopf algebra.

The quantum super algebra $U_q(\mathfrak{gl}(1|1))$ can be realized by introducing a bosonic oscillator b, b^\dagger and a fermionic oscillator a, a^\dagger satisfying

$$[b, b^\dagger] = 1, \{a, a^\dagger\} = 1, \tag{3.21}$$

$$bb^\dagger = N, aa^\dagger = M. \tag{3.22}$$

Then setting¹¹

$$\hat{b} = b \sqrt{\frac{[N]}{N}}, \hat{b}^\dagger = \sqrt{\frac{[N]}{N}} b^\dagger, \hat{a} = a \sqrt{\frac{[M]}{M}}, \hat{a}^\dagger = \sqrt{\frac{[M]}{M}} a^\dagger \tag{3.23}$$

and identifying

$$x = \hat{b}^\dagger \hat{a}, y = \hat{a}^\dagger \hat{b}, \tag{3.24}$$

we can reproduce the commutation relations as in Eqs. (3.12')–(3.15').

The fundamental representation of this algebra is 2-dimensional, which can be read off immediately from (2.16), or from oscillator representation.¹¹ The carried space consists of two states, the bosonic α and the fermionic β .

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |N = 1, M = 0\rangle, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |N = 0, M = 1\rangle, \tag{3.25}$$

$$k = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, l = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.26}$$

Instead of the $\mathfrak{gl}(2)$ case in which $\underline{2} \otimes \underline{2} = \underline{3}_s \oplus \underline{1}_a$, we have

$$\underline{2} \otimes \underline{2} = \underline{2}_s \oplus \underline{2}_a, \tag{3.27}$$

simply because the product of two fermionic states is antisymmetric, $\beta\beta' = -\beta'\beta$, in contrast with the product of two bosonic state being symmetric. In this way we can show that all the irreducible representations are 2-dimensional, which can be corroborated by the nilpotency of operators x and y , i.e., $x^2 = y^2 = 0$. Equation (3.27) is closely related to the fact that the characteristic polynomial of the “non-standard” BGR takes the form

$$(\check{R} - q)^2 (\check{R} + q^{-1})^2 = 0 \tag{3.28}$$

comparing to that of the Jones polynomial

$$(\check{R} - q)^3 (\check{R} + q^{-1}) = 0. \tag{3.29}$$

All the results presented here are similar to those obtained in Ref. 10. The only difference occurs in the signs on the right-hand side of Eqs. (3.4b), (3.5b), (3.6b), and (3.7b) and the anticommutation or commutation bracket in Eq. (3.9). These differ-

ences come from the η factors in Eq. (2.20), which is the character of the supersymmetry.

4. Further Results and Discussions

The analysis of the two-state “non-standard” solution to BGR can be generalized to more general cases. For example, for three-state model, the standard solution to BGR takes the form

$$\begin{aligned} \check{R}_{11}^{11} = \check{R}_{22}^{22} = \check{R}_{33}^{33} = q, \quad \check{R}_{21}^{21} = \check{R}_{31}^{31} = \check{R}_{32}^{32} = q - q^{-1}, \\ \check{R}_{12}^{21} = \check{R}_{21}^{12} = \check{R}_{13}^{31} = \check{R}_{31}^{13} = \check{R}_{23}^{32} = \check{R}_{32}^{23} = 1, \\ \check{R}_{ik}^{jl} = 0, \text{ otherwise} \end{aligned} \tag{4.1}$$

which can be shown being associated with the quantum universal enveloping algebra $U_q(\mathfrak{gl}(3))$, while the “non-standard” solution

$$\begin{aligned} \check{R}_{11}^{11} = \check{R}_{22}^{22} = q, \quad \check{R}_{33}^{33} = -q^{-1}, \quad \check{R}_{21}^{21} = \check{R}_{31}^{31} = \check{R}_{32}^{32} = q - q^{-1}, \\ \check{R}_{12}^{21} = \check{R}_{21}^{12} = \check{R}_{13}^{31} = \check{R}_{31}^{13} = \check{R}_{23}^{32} = \check{R}_{32}^{23} = 1, \\ \check{R}_{ik}^{jl} = 0 \text{ otherwise} \end{aligned} \tag{4.2}$$

can be proved to be connected with the quantum universal enveloping algebra $U_q(\mathfrak{gl}(2|1))$ by a tedious but completely similar analysis. Similar results have been given by Kulish *et al.* in Ref. 7.

For four-state model, it can be shown that the “standard” solution to BGR

$$\begin{aligned} \check{R}_{ii}^{ii} = q (i = 1, 2, 3, 4), \quad \check{R}_{ij}^{ik} = q - q^{-1} (i > k), \\ \check{R}_{ik}^{ki} = \check{R}_{ki}^{ik} = 1 (i \neq k), \quad \check{R}_{ik}^{jl} = 0 \text{ otherwise} \end{aligned} \tag{4.3}$$

determines the quantum universal enveloping algebra $U_q(\mathfrak{gl}(4))$. These are two “non-standard” solutions. One of them is given by

$$\begin{aligned} \check{R}_{ii}^{ii} = q (i = 1, 2, 3), \quad \check{R}_{44}^{44} = -q^{-1} (i > k), \\ \check{R}_{ik}^{ik} = q - q^{-1} (i > k) \\ \check{R}_{ik}^{ki} = \check{R}_{ki}^{ik} = 1 (i \neq k), \quad \check{R}_{ik}^{jl} = 0 \text{ otherwise} \end{aligned} \tag{4.4}$$

which can be shown to be related to the quantum universal enveloping algebra $U_q(\mathfrak{gl}(3|1))$ in a way completely similar to the last section. The other “non-standard” solution to BGR takes the form

$$\begin{aligned}
\check{R}_{ii}^{ii} &= q(i=1,2), \check{R}_{jj}^{jj} = -q^{-1}(j=3,4), \\
\check{R}_{ik}^{ik} &= q - q^{-1}(i > k) \\
\check{R}_{34}^{43} &= \check{R}_{43}^{34} = -1, \\
\check{R}_{ik}^{ki} &= \check{R}_{ki}^{ik} = 1(i \neq k, i, k \neq 3, 4), \\
\check{R}_{ik}^{ji} &= 0 \text{ otherwise.}
\end{aligned} \tag{4.5}$$

Following similar procedure, one can show that this solution yields the quantum universal enveloping algebra $U_q(\mathfrak{gl}(2|2))$. Details of these analysis will be given elsewhere.¹²

It is not difficult to realize that all these arguments can be generalized to BGR solutions corresponding to A_l .⁹ Among these solutions, a series of "standard" ones determines a series of quantum enveloping algebras $U_q(\mathfrak{gl}(l))$, and the "non-standard" series gives quantum universal enveloping algebras $U_q(\mathfrak{gl}(m|n))$, with $m+n=l$.

As for the BGR's corresponding to B, C, D series, things turn out to be much complicated. Special discussion is needed to these cases.¹² But anyway, it seems to us that we can draw the conclusion that quantum Lie superalgebras do appear in the "non-standard" solutions of BGR. On one hand, these "non-standard" solutions are peculiar since they have behavior different from that of "standard" ones with which we are familiar. On the other hand, they are not so peculiar since they have Lie superalgebras as their classical limit with which though we are a little unfamiliar. Undoubtedly a detailed analysis on classical r -matrices corresponding to these Lie superalgebras is very important. We must also emphasize here that not all the "non-standard" BGRs are recognized to be associated with super algebras, especially for B, C, D series. Further investigation is needed for these would-be non-supersymmetric "non-standard" solutions to BGR relations.

Note Added

After sending this manuscript, we learned that, for some concrete examples¹³⁻¹⁶ the Z_2 graded Yang-Baxter equations are also discussed by several groups of authors, and the connection of the simplest "non-standard" braid group representation (1.2) to superalgebras are also obtained using approaches different from ours. We are indebted to the referee for calling our attention to Ref. 13.

References

1. V. G. Drinfeld, *Doklady Akad. Nauk SSSR* **283** (1985) 1060; Talk at ICM-86, Berkeley 1986, Vol. I, pp. 798-820; M. Jimbo, *Lett. Math. Phys.* **10** (1985) 247; *Commun. Math. Phys.* **102** (1986) 537; P. Kulish, and N. Reshetekhin, *Zap Nach Semin LOMI preprint* **101** (1981) 101.
2. C. N. Yang, *Phys. Rev. Lett.* **19** (1967) 1312; R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).
3. L. D. Faddeev, *Les Houches Lectures* (North Holland, 1982).
4. A. Zamolodchikov and Ab. Zamolodchikov, *Ann. Phys.* **120** (1979) 252; H. J. de Vega, *Int. J. Mod. Phys. A* **4** (1989) 2371.

5. G. Moore and N. Seiberg, *Nucl. Phys.* **B313** (1989) 16; *Commun. Math. Phys.* **123** (1989) 177; L. Alvarez-Gaumé, C. Gomez, and G. Sierra, *Phys. Lett.* **B220** (1989) 142; E. Witten, *Commun. Math. Phys.* **121** (1989) 351.
6. N. Yu. Reshetikhin, LOMI preprint E-4-87, E-17-87, 1987 Leningrad.
7. M. Chaichian and P. P. Kulish, *Phys. Lett.* **B234** (1990) 72; T. Hayashi, *Commun. Math. Phys.* **127** (1990) 129; M. Chaichian, P. Kulish, and J. Lukierski, *Phys. Lett.* **B237** (1990) 401; P. Kulish, and N. Yu. Reshetikhin, *Lett. Math. Phys.* **18** (1989) 143.
8. L. Faddeev, N. Reshetikhin, and L. Takhtajan, LOMI preprint E-14-87, 1987 Leningrad; *Braid Group, Knot Theory and Statistical Mechanics*, eds. C. N. Yang and M. L. Ge (World Scientific, 1990), p. 97.
9. H. C. Lee, M. Couture, and N. C. Schmeing, Chalk River preprint 1989; H. C. Lee, *Proc. NATO ARW on Phys. and Geom.*, Lake Takoe, July 1989; M. L. Ge, L. Y. Wang, K. Xue, and Y. S. Wu, *Int. J. Mod. Phys. A4* (1989) 3351; M. L. Ge and K. Xue, ITP-SB-90-20; M. L. Ge, Y. S. Wu and K. Xue ITP-SB-90-02.
10. N. H. Jing, M. L. Ge, and Y. S. Wu, ITP-SB-90-21.
11. X. C. Song, *J. Phys. A-Math. Gen.* **23** (1990) L821.
12. L. Liao and X. C. Song, in preparation.
13. T. Deguchi and Y. Akutsu, *J. Phys. A-Math. Gen.* **23** (1990) 1861.
14. L. H. Kauffman and H. Saluer, Fermi Lab. preprint EF1 90-42; H. Saluer, *Nucl. Phys.* **B336** (1990) 363.
15. J. Avan and M. Talon, PAR LP THE 90-30.
16. W. B. Schmidke, Ph. D Thesis 1990, LBL-29573.