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Canonical Transformations of q-Bosonic Oscillators[1]

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Abstract

Within the framework of R matrix of quantum group, canonical q-transformations for bosonic q-oscillators are presented. The commutation relations are covariant under quantum group $Sp_q(2n)$.

Quantum groups and quantum universal enveloping algebras,[1–3] originated in the study of quantum inverse scattering methods,[4] have attracted many extensive studies. One aim of these studies is to find out whether quantum groups may play some roles in physics as the ordinary symmetry groups play. To this end, there have been some discussions about the deformation of harmonic oscillator which has essential application in physics.[6–8] In this letter, we discuss the q-deformed version of Bogoliubov transformations of n pairs of q-bosonic oscillators, and we show that q-deformed quantum commutation relations are covariant under quantum group $Sp_q(2n)$.

We first briefly describe the results for the usual case of a system with n pairs of creation and annihilation operators. The commutation relations of the operators are

$$[b_i, b_j^\dagger] = 0\,, \quad [b_i^\dagger, b_j^\dagger] = 0\,, \quad [b_i, b_j^\dagger] = \delta_{ij}\,, \tag{1}$$

where $i, j = 1, \ldots, n$. According to the idea of Bogoliubov transformation,[5] these relations are required to be covariant under the following infinitesimal transformations

$$\begin{aligned} b_i \to b_i' &= b_i - i(\alpha_{ik} b_k + \beta_{ik} b_k^\dagger)\,, \\ b_i^\dagger \to b_i'^\dagger &= b_i^\dagger - i(\mu_{ik} b_k + \nu_{ik} b_k^\dagger) \end{aligned} \tag{2}$$

(the convention of summation is understood unless otherwise stated) where the infinitesimal parameters α_{ik}, β_{ik} etc. satisfy $\beta_{ik} = \beta_{ki}$, $\nu_{ik} = \nu_{ki}$ and $\alpha_{ij} = -\mu_{ji}$. It is easy to show that the infinitesimal transformations and the finite transformations induced constitute the group $Sp(2n)$ and that the bilinear form

$$\Lambda \equiv b_i b_i^\dagger - b_i^\dagger b_i = n \tag{3}$$

is an invariant. For the later convenience, let us re-order the operators as follows:

$$y^i = b_{n+1-i}^\dagger\,, \quad y^{n+i} = b_i\,, \qquad i = 1, \ldots, n. \tag{4}$$

Thus, the commutation relations (1) may be re-written as

$$(I_{kl}^{ij} - P_{kl}^{ij}) y^k y^l = \delta^{ij'}\,, \tag{5}$$

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where the unit matrix $I^{ij}_{kl} = \delta^i_k \delta^j_l$, the permutation matrix $P^{ij}_{kl} = \delta^i_l \delta^j_k$ and $j' = 2n + 1 - j$. The canonical transformations now take a more similar form

$$y'^i = \sum_{j=1}^{2n} t^i_j y^j \,, \tag{6}$$

and the matrix t^i_j satisfies

$$\sum_{i,j}^{2n} t^i_k t^j_l g^{kl} = g^{ij} \,, \tag{7}$$

with invariant metric $g^{ij} = \delta^{ij'}$. Now we discuss the case of q-deformed oscillators.[6] Let us denote b_i, $b^\dagger_i$'s deformed counterpart as $\hat{b}^i$, $\hat{b}^{i\dagger}$. These annihilation operators satisfy the deformed commutation relations

$$\hat{b}^i \hat{b}^j - q \hat{b}^j \hat{b}^i = 0 \qquad (i < j) \,, \tag{8}$$

where q is a complex number. By means of the R-matrix for the quantum group $GL_q(n)$ it is easy to show that relations (8) are equivalent to

$$S^{ij}_{kl} \hat{b}^k \hat{b}^l \equiv (I^{ij}_{kl} - q^{-1} \check{R}^{ij}_{kl}) \hat{b}^k \hat{b}^l = 0 \,, \tag{9}$$

where

$$\begin{aligned} \check{R}^{ij}_{kl} &= \delta^j_k \delta^i_l (1 + (q-1)\,\delta^{ij}) + (q - q^{-1})\,\delta^i_k \delta^j_l \theta_{ji} \,, \\ \theta_{ji} &= \begin{cases} 1 & \text{if } j > i \,, \\ 0 & \text{otherwise} \,. \end{cases} \end{aligned} \tag{10}$$

Taking the hermitian conjugate of Eqs (10), we get the q-deformed commutation relations among $\hat{b}^\dagger$'s

$$\hat{b}^\dagger_j \hat{b}^\dagger_i - q^{-1} \hat{b}^\dagger_i \hat{b}^\dagger_j = 0 \qquad (i < j) \,, \tag{11}$$

where q is taken to be the root of one, i.e. $q^* = q^{-1}$. For a consistent deformed algebra, we assume the commutation relations between $\hat{b}$'s and $\hat{b}^\dagger$'s as follows:

$$\hat{b}^\dagger_j \hat{b}^i = f^i_j + C^{il}_{jk} \hat{b}^k \hat{b}^\dagger_l \,, \tag{12}$$

where f^i_j and C^{il}_{jk} are numerical coefficients. As pointed out in Ref. [7], these oscillators are not likely independent, i.e. there are only nontrivial deformed version of permutation matrix P in generalizing relations (5) as we show in the following. The relations (9) should still be valid when applied on a Fock state $F(\hat{b}, \hat{b}^\dagger)|0\rangle$

$$S^{ij}_{kl} \hat{b}^k \hat{b}^l F(\hat{b}, \hat{b}^\dagger)|0\rangle = 0 \,. \tag{13}$$

Multiplying Eq. (13) from the left by $\hat{b}^\dagger_m$ and using the relations (12), we have

$$\begin{aligned} S^{ij}_{kl} \hat{b}^\dagger_m \hat{b}^k \hat{b}^l F(\hat{b}, \hat{b}^\dagger)|0\rangle &= S^{ij}_{kl} (f^k_m \delta^l_n + C^{kp}_{mn} f^l_p)\, \hat{b}^n F(\hat{b}, \hat{b}^\dagger)|0\rangle \\ &+ S^{ij}_{kl} C^{kp}_{mb} C^{ld}_{pe} \hat{b}^b \hat{b}^e (\hat{b}^\dagger_d F(\hat{b}, \hat{b}^\dagger)|0\rangle) = 0 \,. \end{aligned} \tag{14}$$

Since $F(\hat{b}, \hat{b}^\dagger)|0\rangle$ and $\hat{b}^\dagger_d F(\hat{b}, \hat{b}^\dagger)|0\rangle$ are different states of the Fock space, we get the consistency conditions separately

$$S^{ij}_{kl} (f^k_m \delta^l_n + C^{kp}_{mn} f^l_p) = 0 \,, \tag{15}$$

$$S^{ij}_{kl} C^{kp}_{mb} C^{ld}_{pe}\, \hat{b}^b \hat{b}^e = 0 \,. \tag{16}$$

Using Eqs (9), relations (16) can be written as

$$(\check{R}^{ij}_{kl} C^{kp}_{mb} C^{ld}_{pe} - C^{ia}_{ml} C^{jd}_{ak} \check{R}^{lk}_{be})\, \hat{b}^b \hat{b}^e = 0\,. \tag{17}$$

This equation is satisfied if the Yang–Baxter equation

$$\check{R}^{ij}_{kl} C^{kp}_{mb} C^{ld}_{pe} = C^{ia}_{ml} C^{jd}_{ak} \check{R}^{lk}_{be} \tag{18}$$

holds. It is now obvious that, if we take

$$f^i_j = \delta^i_j f_j\,, \qquad C^{ij}_{kl} = q f_k \check{R}^{ij}_{kl} f_j^{-1}\,, \tag{19}$$

the consistency conditions are satisfied (k is not summed). With solutions (19) and the relation between $\check{R}$ matrices of $GL_q(n)$ and $Sp_q(2n)$ (Ref. [9]), we can put relations (9), (11) and (12) into a more compact form as follows:

$$(I^{ij}_{kl} - q^{-1} \check{Q}^{ij}_{kl})\, \hat{y}^k \hat{y}^l = G^{ij}\,, \tag{20}$$

where

$$\hat{y}^i \equiv \hat{b}^\dagger_{n+1-i}\,, \qquad \hat{y}^{n+i} \equiv \hat{b}^i \qquad (i = 1, \cdots, n)\,, \tag{21}$$

$$\check{Q}^{ij}_{kl} \equiv F_i F_j \check{\mathbf{R}}^{ij}_{kl} F^k F^l\,, \qquad G^{ij} \equiv F_i F_j C^{ij}\,, \tag{22}$$

$$F^k \equiv (q^{-n+2-k} f_{n+1-k})^{-1/2}\,, \quad F^{n+k} \equiv (q^{-2n+1+k} f_k)^{-1/2}\,, \quad (k = 1, \cdots, n) \tag{23}$$

and

$$F_i = (F^i)^{-1}\,. \tag{24}$$

Here

$$\begin{aligned} \check{\mathbf{R}}^{ij}_{kl} = {} & q \delta^j_k \delta^i_l \delta^{ij} + q^{-1} \delta^j_k \delta^i_l \delta^{ij'} + (q - q^{-1})\, \theta_{ji} \delta^j_l \delta^i_k \\ & - (q - q^{-1})\, \theta_{jk} q^{\rho_j - \rho_k} \epsilon_j \epsilon_k \delta^{ij'} \delta^{kl'} + (1 - \delta^{ij})(1 - \delta^{kl'})\, \delta^j_k \delta^i_l \end{aligned} \tag{25}$$

and

$$C^{ij} = \delta^{ij'} q^{\rho_j}\,. \tag{26}$$

$\check{\mathbf{R}}^{ij}_{kl}$ in Eq. (25) is the $\check{R}$ matrix for $Sp_q(2n)$. The indices i, j etc. range from 1 to $N = 2n$. And

$$\rho_k = (n, n-1, n-2, \ldots, 1, -1, -2, \ldots, -n)\,, \tag{27}$$

$$\epsilon_k = (1, 1, \ldots, 1, -1, -1, \ldots, -1)\,. \tag{28}$$

It is now obvious that relations (20) are covariant under the action of a linear transformation

$$\hat{y}^i \longrightarrow \hat{y}'^i = T^i_j \hat{y}^j\,, \tag{29}$$

where the matrix elements T^i_j are assumed to commute with $\hat{y}^i$, provided that

$$\check{\mathbf{R}}^{ij}_{kl} T^k_a T^l_b = T^i_k T^j_l \check{\mathbf{R}}^{kl}_{ab}\,, \tag{30}$$

$$T^i_k T^j_l J^{kl} = J^{ij}\,. \tag{31}$$

These are just the conditions for matrices Ts to belong to the quantum group $Sp_q(2n)$. All the deformed relations will go back to the ordinary ones in the limit $q \longrightarrow 1$.

To sum up, we present canonical q-transformations of n pairs of q-oscillators, which is a generalization of Bogoliubov transformations in the q-deformed case. It is obvious that in this scheme the canonical q-transformations cannot be set up with truly independent q-oscillators, as reported in Ref. [8]. In the above discussions, the transformation matrix elements are assumed to commute with operators. This is different from the treatment in Ref. [8]. It seems interesting to investigate the harmonic realization of the canonical transformations presented in this letter.

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