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Spinor and Oscillator Representations of Quantum Enveloping Algebras of Type B_n, C_n and D_n[1]

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Abstract

With the help of the q-deformed bosonic and fermionic oscillation operators, which can be constructed from the ordinary ones, the quantum enveloping algebras of the classical Lie algebras B, C and D are written down explicitly. Under these representations the highest roots are given.

I. Introduction

During recent years, the quantum groups or the q-deformation of the universal enveloping algebras[1–3] cause more and more interest from both physical and mathematical point of views. More recently, a new realization has been proposed[4] for the simplest example $U_q(\mathrm{su}(2))$, in which the generators $\tilde{J}_0$ and $\tilde{J}_\pm$ are expressed as the bilinear form of the q-deformed boson operators $\tilde{a}_i$ and $\tilde{a}_i^\dagger (i = 1,2)$ similar to the Schwinger form of the angular momentum operators in su(2) algebra. It has been shown that the q-oscillator operators $\tilde{a}_i$ and $\tilde{a}_i^\dagger$ can be constructed from the ordinary ones a_i and $a_i^\dagger$[5]. The similar consideration has also been applied to some concrete examples for other quantum algebras or quantum superalgebras[6]. Here we give the systematic results about quantum algebras B, C and D as well as the way to specify the highest root for each algebra, which is important in constructing the quantum parametrized R-matrix[7] for quantum Yang-Baxter equation. This procedure can be generalized to obtain the oscillator form of the Lie superalgebras[8].

II. Oscillator Representation of Classical Lie Algebras and Clifford Algebra

It is well known that $\mathrm{su}(n) \sim A_{n-1}$ algebra can be realized by introducing n-independent bosonic or fermionic oscillators

$$E_{ij} = c_i^\dagger c_j \,, \tag{1}$$

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where for bosonic case $c_i = b_i$, $[b_i, b_j^\dagger] = \delta_{ij}$ with other brackets vanishing; and for fermionic case $c_i = a_i$, $\{a_i, a_j^\dagger\} = \delta_{ij}$ with other anti-commutation brackets vanishing. For both of these identifications one always has

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} , \tag{2}$$

which defines the su(n) algebra. An important feature of Eq. (1) is that the total "particle number" is always conserved for A_{n-1} algebra.

It is also known that[9] sp($2n$) $\sim C_n$ algebra can also be described by n pairs of bosonic operators b_i and $b_i^\dagger$. The generators of C_n in Chevalley bases can be identified as follows:

$$h_i = b_i^\dagger b_i - b_{i+1}^\dagger b_{i+1} , \qquad e_i = b_i^\dagger b_{i+1} , \qquad f_i = b_{i+1}^\dagger b_i \quad \text{for } i = 1 \text{ to } n-1 , \tag{3}$$

and

$$h_n = b_n^\dagger b_n + \frac{1}{2} , \qquad e_n = -\frac{1}{2} b_n^\dagger b_n^\dagger , \qquad f_n = \frac{1}{2} b_n b_n . \tag{4}$$

This yields the commutation rules

$$\begin{aligned} &[h_i, e_j] = a_{ij} e_j , \qquad [h_i, f_j] = -a_{ij} f_j , \\ &[e_i, f_j] = \delta_{ij} h_i , \qquad [h_i, h_j] = 0 , \qquad i, j = 1, 2, \cdots, n , \end{aligned} \tag{5}$$

where a_{ij} is the Cartan matrix corresponding to sp($2n$) algebra, $a_{ii} = 2$, $a_{i,j} = -1$ ($i = j \pm 1, i, j \leq n-1$), $a_{n-1,n} = -2$, $a_{n,n-1} = -1$ and zero elsewhere. Another important property is that, the set of operators $\{b_1^\dagger, b_2^\dagger, \cdots, b_n^\dagger, b_n, \cdots, b_1\}$ forms the $2n$-dimensional vector representation of C_n, i.e.,

$$\begin{aligned} &[h_i, b_j^\dagger] = b_i^\dagger \delta_{i,j} - b_{i+1}^\dagger \delta_{i+1,j} , \quad [h_i, b_j] = -b_i \delta_{i,j} + b_{i+1} \delta_{i+1,j} , \\ &[e_i, b_j^\dagger] = b_i^\dagger \delta_{i+1,j} , \qquad [e_i, b_j] = -\delta_{i,j} b_{i+1} , \\ &[f_i, b_j^\dagger] = b_{i+1}^\dagger \delta_{i,j} , \qquad [f_i, b_j] = -\delta_{i+1,j} b_i , \end{aligned} \tag{6}$$

$$\begin{aligned} &[h_n, b_j^\dagger] = b_n^\dagger \delta_{nj} , \qquad [e_n, b_j^\dagger] = 0 , \qquad [f_n, b_j^\dagger] = b_n \delta_{nj} , \\ &[h_n, b_j] = b_n \delta_{nj} , \qquad [e_n, b_j] = b_n^\dagger \delta_{nj} , \qquad [f_n, b_j] = 0 . \end{aligned} \tag{7}$$

For C_n case, the "particle number" is conserved only modulo two. So b_i and $b_j^\dagger$ are in the same irreducible representation.

The orthogonal algebras o(N) can be realized through the Clifford algebra, which is a set of matrices satisfying the anticommutation relations

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB} , \qquad A, B = 1, 2, \cdots, N . \tag{8}$$

Then the generators of o(N) can be constructed as

$$M_{AB} = \frac{1}{4i} [\Gamma_A, \Gamma_B] . \tag{9}$$

It is easy to show that

$$[M_{AB}, \Gamma_C] = i(\delta_{AC} \Gamma_B - \delta_{BC} \Gamma_A) , \tag{10}$$

and

$$[M_{AB},\, M_{CD}] = i(\delta_{AC}M_{BD} - \delta_{BC}M_{AD}) + i(\delta_{AD}M_{CB} - \delta_{BD}M_{CA})\,. \tag{11}$$

This means that $\{M_{AB}\}$ is the standard form for rotation generators and Γ's are a set of tensors transforming according to the N-dimensional vector representation of $o(N)$..

The Clifford algebra can be put into the operator form by introducing a set of fermionic oscillators. Consider n pairs of creation and annihilation operators with the commutation relations

$$\{a_i,\, a_j^\dagger\} = \delta_{ij}\,, \qquad \{a_i,\, a_j\} = 0 = \{a_i^\dagger,\, a_j^\dagger\}\,, \tag{12}$$

Γ_A $(A = 1 \text{ to } 2n)$ identified as

$$\begin{aligned} \Gamma_1 &= a_1 + a_1^\dagger\,, & \Gamma_2 &= i(a_1 - a_1^\dagger)\,, \\ \Gamma_3 &= a_2 + a_2^\dagger\,, & \Gamma_4 &= i(a_2 - a_2^\dagger)\,, \\ &\cdots\cdots\,, & &\cdots\cdots\,, \\ \Gamma_{2n-1} &= a_n + a_n^\dagger\,, & \Gamma_{2n} &= i(a_n - a_n^\dagger) \end{aligned} \tag{13}$$

can be easily proved to satisfy relation (8). Each pair of operators a_k and $a_k^\dagger$ acts on a two-dimensional space which is also the representation space of Γ_{2k-1} and Γ_{2k}. It can be easily shown that

$$\Gamma_{2k-1}\Gamma_{2k} = i(-1)^{M_k+1}\,, \qquad M_k = a_k^\dagger a_k\,. \tag{14}$$

The total space operators Γ_A $(A = 1 \text{ to } 2n)$ acting upon are 2^n dimensional. One can construct one more operator Γ_{2n+1} operating on the same space

$$\Gamma_{2n+1} \equiv a_0 = (-1)^M\,, \qquad M = \sum_{k=1}^{n} M_k \tag{15}$$

with the properties

$$\Gamma_{2n+1} = (i)^n\Gamma_1\Gamma_2\cdots\Gamma_n\,, \qquad a_0^2 = 1 \tag{16}$$

and

$$\{a_0, a_k\} = 0\,, \qquad \{a_0,\, a_k^\dagger\} = 0\,, \tag{17}$$

which follow from relations

$$(1 - M)a_k = a_k M\,, \qquad M a_k^\dagger = a_k(1 - M)\,.$$

By an appropriate diagonalization for $\{M_{AB}\}$, the Chevalley generators of $o(N)$ can be written as follows. The first $(n-1)$ simple roots and coroots are expressed as

$$h_k = a_k^\dagger a_k - a_{k+1}^\dagger a_{k+1}\,, \qquad e_k = a_k^\dagger a_{k+1}\,, \qquad f_k = a_{k+1}^\dagger a_k \tag{18}$$

for $1 \le k \le n-1$, whereas the last ones are given by

$$h_n = a_{n-1}^\dagger a_{n-1} + a_n^\dagger a_n - 1\,, \qquad e_n = a_{n-1}^\dagger a_n^\dagger\,, \qquad f_n = a_n a_{n-1} \tag{19}$$

for $D_n \sim o(2n)$, and

$$h_n = 2a_n^\dagger a_n - 1\,, \qquad e_n = a_n^\dagger a_0\,, \qquad f_n = a_0 a_n \tag{20}$$

for $B_n \sim o(2n+1)$.

For D_n algebra, where only the first (2n) of the Γ's are concerned with, all the generators in Eqs. (18) and (19) conserve the total "particle number" M modulo two. The last one of the Γ's now commutates with all the generators. Thus, as we suspected, the total 2^n-dimensional spinor space is not irreducible, but falls apart into two 2^{n-1}- dimensional representations. For $a_0 = 1$ it is the spinor representation $D^{(n)}$; for $a_0 = -1$ it gives $D^{(n-1)}$, the other spinor representation.

For B_n algebra, all the Γ's appear in generators Eqs. (18) and (20). The total "particle number" M is no longer conserved since e_n and f_n contain odd number of a (or $a^\dagger$)'s. In this case, the total 2^n-dimensional space is irreducible. This is the spinor representation $D^{(n)}$ of B_n.

As stated above, the Γ's used in constructing the generators form the vector representation $D^{(1)}$ of the orthogonal algebras. It is $(2n)$ dimensional for $D_n \sim o(2n)$, namely

$$(a_1^\dagger,\, a_2^\dagger,\, \cdots,\, a_{n-1}^\dagger,\, a_n^\dagger,\, a_n,\, a_{n-1},\, \cdots,\, a_2,\, a_1)$$

and $(2n+1)$ dimensional for $B_n \sim o(2n+1)$, i.e.,

$$\left(a_1^\dagger,\, a_2^\dagger,\, \cdots,\, a_n^\dagger,\, \frac{a_0}{\sqrt{2}},\, -a_n,\, \cdots,\, -a_2,\, -a_1\right) .$$

It can be shown that

$$\begin{aligned}
&[e_i,\, a_j^\dagger] = a_i^\dagger \delta_{i+1,j}\,, && [e_i,\, a_j] = -\delta_{i,j} a_{i+1}\,,\\
&[f_i,\, a_j^\dagger] = a_{i+1}^\dagger \delta_{i,j}\,, && [f_i,\, a_j] = -\delta_{i+1,j} a_i\,,\\
&[h_i,\, a_j^\dagger] = a_i^\dagger \delta_{i,j} - a_{i+1}^\dagger \delta_{i+1,j}\,, && [h_i,\, a_j] = -a_i \delta_{i,j} + a_{i+1}\delta_{i+1,j}
\end{aligned} \tag{21}$$

for $i,\, j = 1$ to $n-1$, and

$$\begin{aligned}
&[e_n,\, a_k^\dagger] = 0\,, && [e_n,\, a_k] = \delta_{nk} a_{n-1}^\dagger - \delta_{n-1,k} a_n^\dagger\,,\\
&[f_n,\, a_k] = 0\,, && [f_n,\, a_k^\dagger] = a_n \delta_{n-1,k} - a_{n-1}\delta_{n,k}\,,\\
&[h_n,\, a_k^\dagger] = a_{n-1}^\dagger \delta_{n-1,k} + a_n^\dagger \delta_{n,k}\,, && [h_n,\, a_k] = -a_n \delta_{nk} - \delta_{n-1,k} a_{n-1}
\end{aligned} \tag{22}$$

for D_n, and

$$\begin{aligned}
&[e_n,\, a_k^\dagger] = 0\,, && [e_n,\, a_k] = -\delta_{n,k} a_0 + 2\delta_{0,k} a_n^\dagger\,,\\
&[f_n,\, a_k^\dagger] = a_0 \delta_{n,k}\,, && [f_n,\, a_k] = -2\delta_{0,k} a_n\,,\\
&[h_n,\, a_k^\dagger] = 2a_n^\dagger \delta_{n,k}\,, && [h_n,\, a_k] = -2a_n \delta_{n,k}
\end{aligned} \tag{23}$$

for B_n.

The basis of the vector representation for C_n, D_n and B_n and their transition under the action of the simple step generators are represented by the diagrams in Figs. 1, 2 and 3.

These representations are very important in constructing the oscillator formalism of the Lie superalgebras[8]. Notice that under the action of the subalgebra (e_k, h_k, f_k) all the states in $D^{(1)}$ are split into singlets or doublets for C_n and D_n. Only for B_n algebra, we have triplet structure in $D^{(1)}$, e.g., $(a_n^\dagger, a_0/\sqrt{2}, -a_n)$ being a triplet under the action of subalgebra (e_n, h_n, f_n).

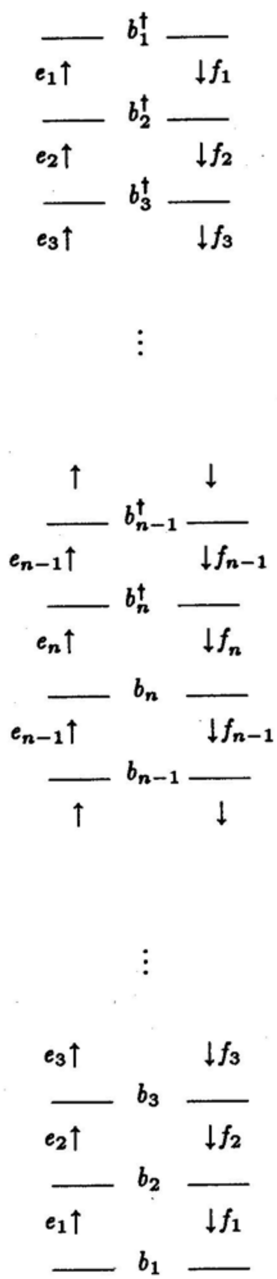

Fig. 1. Vector rep. of C_n.

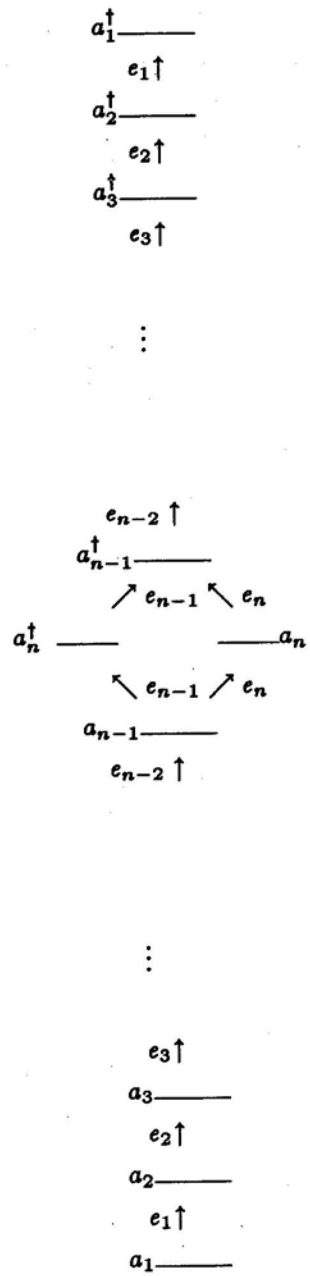

Fig. 2. Vector rep. of D_n.

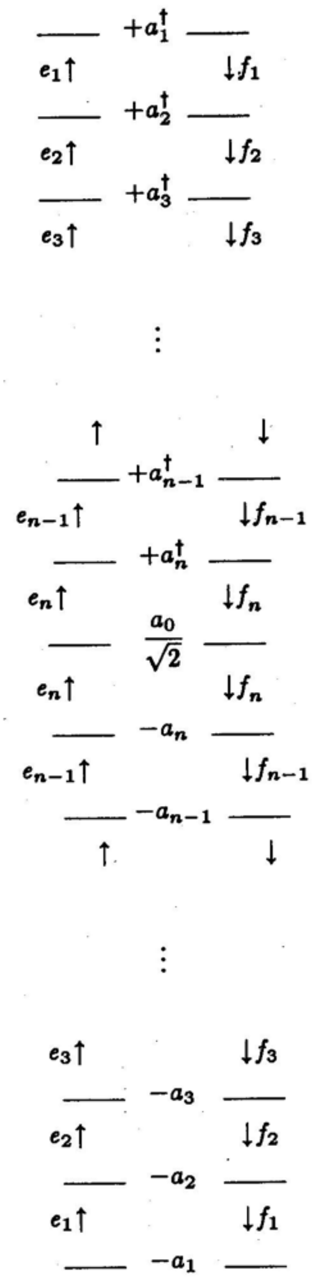

Fig. 3. Vector rep. of B_n.

III. Oscillator Representation of Quantum Enveloping Algebra $U_q(C_n)$

Now we turn to discuss the oscillator representation of the quantum enveloping algebras. The first step has been taken by several authors[4]. In their articles, at first $U_q(su(2))$ and then

$U_q(su(n))$ are realized by introducing q-analogues of quantum harmonic oscillator. Taking over their q-analogue of bosonic operator, we realize quantum enveloping algebra of type C_n in a minimum way which may be more useful in some cases.

The quantum enveloping algebra $U_q(g)$ corresponding to complex simple Lie algebra g is given by Jimbo[3]. Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix and $\{\alpha_i\}_{1\le i\le n}$, $\{h_i\}_{1\le i\le n}$ the simple roots and coroots such that $\langle h_i, \alpha_j\rangle = a_{ij}$. For a nonzero parameter t we write $t_i = t^{(\alpha_i|\alpha_i)/2}$ so that $t_i^{a_{ij}} = t^{(\alpha_i|\alpha_j)} = t_j^{a_{ij}}$, where $(\ |\ )$ denotes the invariant inner product in $\eta^* = \oplus C\alpha_i$. Thus the quantum enveloping algebra $U_q(g)$ is generated by the following relations:

$$k_i = t_i^{h_i}\,, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1\,, \qquad [k_i, k_j] = 0\,, \tag{24a}$$

$$k_i e_j k_i^{-1} = t_i^{a_{ij}} e_j\,, \qquad k_i f_j k_i^{-1} = t_i^{-a_{ij}} f_j\,, \tag{24a}$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{t_i^2 - t_i^{-2}}\,, \tag{24c}$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{t_i^2} e_i^{1-a_{ij}-\nu} e_j e_i^\nu = 0 \qquad (i \neq j)\,, \tag{24d}$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{t_i^2} f_i^{1-a_{ij}-\nu} f_j f_i^\nu = 0 \qquad (i \neq j)\,. \tag{24e}$$

Here t is an arbitrary parameter, and the symbol $\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{bmatrix} m \\ m-n \end{bmatrix}_t$ is defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \dfrac{(t^m - t^{-m})(t^{m-1} - t^{-m+1})\cdots(t^{m-n+1} - t^{-m+n-1})}{(t - t^{-1})(t^2 - t^{-2})\cdots(t^n - t^{-n})} & (m > n > 0)\,, \\ 1 & (n = 0,\, m)\,, \end{cases} \tag{25}$$

and $\begin{bmatrix} m \\ n \end{bmatrix}_t = 0$ otherwise.

For C_n algebra, its Cartan matrix $A = (a_{ij}) = 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i)$ has the following form:

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & 2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & -1 & 0 \\ & & & -1 & 2 & -2 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{pmatrix}, \tag{26}$$

i.e.,

$$\begin{aligned} &a_{ii} = 2\,, \qquad a_{ij} = -1 \qquad (i = j \pm 1\,, \quad i, j \le n-1)\,, \\ &a_{n-1,n} = -2\,, \qquad a_{n,n-1} = -1\,, \qquad a_{ij} = 0 \qquad \text{otherwise}\,. \end{aligned} \tag{27}$$

So we can write Eqs. (24a), (24b) and (24c) in $U_q(C_n)$ as follows:

$$
\begin{aligned}
&[h_i, h_j] = 0 && i, j = 1, 2, \cdots, n, \\
&[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j && i, j = 1, \cdots, n, \\
&[e_i, f_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \equiv \delta_{ij}[h_i] && i, j = 1, \cdots, n-1, \\
&[e_n, f_n] = \frac{q^{2h_n} - q^{-2h_n}}{q^2 - q^{-2}} \equiv [h_n]_{q^2} = \frac{[2h_n]}{[2]}
\end{aligned}
\tag{28}
$$

with $t_1 = t_2 = \cdots = t_{n-1} = q^{1/2}$, $t_n = q$, and $[x] \equiv [x]_q$. If let $q \to 1$, the above relations reduce to the usual definition relations of C_n in Chevalley basis which, together with the oscillator form of the generators, have already been given in Eqs. (3)–(5).

To extrapolate to $q \neq 1$ case, we simply make the following correspondence[5]:

$$
\begin{aligned}
b_i &\to \tilde{b}_i = b_i\sqrt{\frac{[N_i]}{N_i}} = \sqrt{\frac{[N_i+1]}{N_i+1}}\, b_i, \\
b_i^\dagger &\to \tilde{b}_i^\dagger = b_i^\dagger\sqrt{\frac{[N_i+1]}{N_i+1}} = \sqrt{\frac{[N_i]}{N_i}}\, b_i^\dagger,
\end{aligned}
\tag{29}
$$

where $N_i = b_i^\dagger b_i$ is the number operator for i-th oscillator and so

$$\tilde{b}_i^\dagger \tilde{b}_i = [N_i], \qquad \tilde{b}_i \tilde{b}_i^\dagger = [N_i + 1]. \tag{30}$$

Then we obtain

$$h_i \to N_i - N_{i+1}, \qquad 1 \le i \le n-1, \tag{31a}$$

$$h_n \to N_n + \frac{1}{2}, \tag{31b}$$

$$\tilde{e}_i \to \tilde{b}_i^\dagger \tilde{b}_{i+1}, \qquad \tilde{f}_i \to \tilde{b}_{i+1}^\dagger \tilde{b}_i, \qquad 1 \le i \le n-1, \tag{31c}$$

$$\tilde{e}_n \to -\frac{\tilde{b}_n^\dagger \tilde{b}_n^\dagger}{q + q^{-1}}, \qquad \tilde{f}_n \to \frac{\tilde{b}_n \tilde{b}_n}{q + q^{-1}}. \tag{31d}$$

Omitting the lengthy derivation, we instead present some relations which are useful in reproducing Eqs. (28):

$$[N+1] - q^{\pm 1}[N] = q^{\mp N}, \tag{32}$$

$$[N_1][N_2+1] - [N_2][N_1+1] = [N_1 - N_2], \tag{33}$$

$$[N][N+1] - [M][M+1] = [N+M+1][N-M]. \tag{34}$$

In this framework, the highest root e_0 can be written out as

$$e_0 = \frac{1}{[2]^{n-1}\prod_{i=2}^{n} q^{2N_i+2}}(e_n|e_{n-1}, e_{n-1}|e_{n-2}, e_{n-2}|\cdots|e_1, e_1), \tag{35}$$

where

$$(A|B, C) \equiv [[A, B]_{q^2}, C] \tag{36}$$

with the definition

$$[A,\, B]_{q^2} = AB - q^2 BA\,, \tag{37}$$

and recurrence convention is understood when C in Eq. (36) has the similar construction, say $C = (A'|B',\, C')$ and so on.

IV. Spinor Representations of Quantum Enveloping Algebras $U_q(D_n)$ and $U_q(B_n)$

As is known, the definition relations of D_n and B_n are given by

$$[h_i,\, h_j] = 0\,, \qquad i,\, j = 1,\, 2,\, \cdots,\, n\,, \tag{38a}$$

$$[h_i,\, e_j] = a_{ij} e_j\,, \qquad [h_i,\, f_j] = -a_{ij} f_j\,, \tag{38b}$$

$$[e_i,\, f_j] = \delta_{ij} h_j \tag{38c}$$

with

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & 2 & & & \vdots \\ \vdots & & & & & 0 \\ \vdots & & & 2 & -1 & -1 \\ \vdots & & & -1 & 2 & 0 \\ 0 & & & -1 & 0 & 2 \end{pmatrix} \quad \text{for } D_n\,, \tag{39}$$

and

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & 2 & & & \\ \vdots & & & & & \vdots \\ & & & 2 & -1 & 0 \\ \vdots & & & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -2 & 2 \end{pmatrix} \quad \text{for } B_n\,. \tag{40}$$

It is easy to see that the relations (38a)–(38c) are just Eqs. (24a)–(24c) in the $q \to 1$ limit with a_{ij} specified as above. When $q \neq 1$, the relations (24a)–(24c) can be written as: for D_n,

$$[h_i,\, h_j] = 0 \qquad i,\, j = 1,\, 2,\, \cdots,\, n\,, \tag{41a}$$

$$[h_i,\, e_j] = a_{ij} e_j\,, \qquad [h_i,\, f_j] = -a_{ij} f_j\,, \tag{41b}$$

$$[e_i,\, f_j] = \delta_{ij} [h_j]_q \tag{41c}$$

with $t_1 = t_2 = \cdots = t_n = q^{1/2}$;

for B_n,

$$[h_i, h_j] = 0 \qquad i, j = 1, 2, \cdots, n\,, \tag{42a}$$

$$[h_i, e_j] = a_{ij} e_j\,, \qquad [h_i, f_j] = -a_{ij} f_j \tag{42b}$$

$$[e_i, f_j] = \delta_{ij}[h_j]_q \qquad 1 \le i, j \le n-1\,, \tag{42c}$$

$$[e_n, f_n] = [h_n]_{q^{\frac{1}{2}}} = \left[\frac{h_n}{2}\right]_q \Big/ \left[\frac{1}{2}\right]_q \tag{42d}$$

with $t_1 = t_2 = \cdots = t_{n-1} = q^{1/2}$, $t_n = q^{1/4}$.

To regain these relations, we need to find a q-analogue of fermionic operator. To this end, we can use the similar prescription given in Ref. [5] with a slight modification.

From the definition of the number operator, we have

$$a^\dagger a = M\,, \quad a a^\dagger = 1 - M\,, \tag{43}$$

and

$$aM = (1-M)a\,, \quad a^\dagger(1-M) = M a^\dagger\,. \tag{44}$$

Now we define the q-analogue of fermionic operators $\tilde{a}$ and $\tilde{a}^\dagger$ as follows:

$$\tilde{a}_i = a_i \sqrt{\frac{[M_i]}{M_i}} = \sqrt{\frac{[1-M_i]}{1-M_i}}\, a_i\,, \tag{45a}$$

$$\tilde{a}_i^\dagger = a_i^\dagger \sqrt{\frac{[1-M_i]}{1-M_i}} = \sqrt{\frac{[M_i]}{M_i}}\, a_i^\dagger\,. \tag{45b}$$

Then we get

$$\tilde{a}_i^\dagger \tilde{a}_i = [M_i]\,, \quad \tilde{a}_i \tilde{a}_i^\dagger = [1-M_i]\,, \tag{46a}$$

$$\tilde{a}_i^2 = (\tilde{a}_i^\dagger)^2 = 0\,, \tag{46b}$$

equivalently

$$\{\tilde{a}_i, \tilde{a}_i^\dagger\}_q = \tilde{a}_i \tilde{a}_i^\dagger + q \tilde{a}_i^\dagger \tilde{a}_i = q^{M_i}\,, \tag{47a}$$

$$\{\tilde{a}_i, \tilde{a}_i^\dagger\}_{q^{-1}} = \tilde{a}_i \tilde{a}_i^\dagger + q^{-1} \tilde{a}_i^\dagger \tilde{a}_i = q^{-M_i}\,. \tag{47b}$$

Then the relations (41) and (42) can be reproduced with the following identifications:

for D_n,

$$h_i = M_i - M_{i+1}\,, \tag{48a}$$

$$h_n = M_{n-1} + M_n - 1\,, \tag{48b}$$

$$e_i = \tilde{a}_i^\dagger \tilde{a}_{i+1}\,, \qquad f_i = \tilde{a}_{i+1}^\dagger \tilde{a}_i\,, \tag{48c}$$

$$e_n = \tilde{a}_{n-1}^\dagger \tilde{a}_n^\dagger\,, \qquad f_n = \tilde{a}_n \tilde{a}_{n-1}\,; \tag{48d}$$

for B_n,

$$h_i = M_i - M_{i+1} \,, \tag{49a}$$

$$h_n = 2M_n - 1 \,, \tag{49b}$$

$$e_i = \tilde{a}_i^\dagger \tilde{a}_{i+1} \,, \qquad f_i = \tilde{a}_{i+1}^\dagger \tilde{a}_i \,, \tag{49c}$$

$$e_n = \tilde{a}_n^\dagger a_0 \,, \qquad f_n = a_0 \tilde{a}_n \,, \tag{49d}$$

where $a_0 = (-1)^M$ remains to be the same as before. It is not difficult to check that the identifications given in Eqs. (48) and (49) do satisfy the definition relations as in Eqs. (41) and (42). In doing so, besides the relations given in Eqs. (32)–(34), we also use

$$[M_1][M_2] - [1 - M_1][1 - M_2] = [M_1 + M_2 - 1] \,, \tag{50}$$

$$[M] - [1 - M] = \left[M - \frac{1}{2}\right] \Big/ \left[\frac{1}{2}\right] . \tag{51}$$

In this framework, the highest root can be written out as follows:

for B_n,

$$e_0 = (-1)^n \frac{1}{2} q^{-M_2-(n-1)} \langle e_1, e_2; e_2, e_3; \cdots; e_{n-2}, e_{n-1}; e_n \rangle_q \,, \tag{52}$$

where

$$\langle A, B; C \rangle_q \equiv [[A, B]_q, C]_q \tag{53}$$

with $[A, B]_q = AB - qBA$;

for D_n,

$$e_0 = \langle\langle e_1, e_2; e_2, e_3; \cdots; e_{n-2}, e_{n-1}; e_n \rangle\rangle_q \,, \tag{54}$$

where

$$\langle\langle e_i, e_{i+1}; Q \rangle\rangle_q \equiv [[e_i, e_{i+1}]_q q^{-M_{i+1}}, Q]_q q^{-M_{i+2}} \,. \tag{55}$$

The details of these calculations will be published elsewhere.

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