Program Self-Reference in Constructive Scott Subdomains*

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Abstract. Intuitively, a *recursion theorem* asserts the existence of *self-referential programs*. Two well-known recursion theorems are Kleene's Recursion Theorem (krt) and Rogers' Fixpoint Recursion Theorem (fprt). Does one of these two theorems better capture the notion of program self-reference than the other? In the context of the partial computable functions over the natural numbers ($\mathcal{PC}$), fprt is strictly *weaker* than krt, in that fprt holds in any effective numbering of $\mathcal{PC}$ in which krt holds, but *not* vice versa. It is shown that, in this context, the existence of *self-reproducing programs* (a.k.a. *quines*) is assured by krt, but *not* by fprt. Most would surely agree that a self-reproducing program is self-referential. Thus, this result suggests that krt is better than fprt at capturing the notion of program self-reference *in* $\mathcal{PC}$.

A generalization of krt to arbitrary *constructive Scott subdomains* is then given. (For fprt, a similar generalization was already known.) Surprisingly, for some such subdomains, the two theorems turn out to be *equivalent*. A precise characterization is given of those constructive Scott subdomains in which this occurs. For such subdomains, the two theorems capture the notion of program self-reference equally well.

Keywords: numberings, recursion theorems, Scott domains, self-reference, self-reproducing programs.

1 Introduction: krt and fprt

Intuitively, a *recursion theorem* asserts the existence of *self-referential programs*. Two well-known recursion theorems are Kleene's Recursion Theorem (krt) and Rogers' Fixpoint Recursion Theorem (fprt). Does one of these two theorems better capture the notion of program self-reference than the other?

The two theorems are normally stated in the context of the partial computable functions over the natural numbers ($\mathcal{PC}$). We give the formal statements of the two theorems following some necessary definitions.

* This is an expanded version of [CM09b].

Let $\mathbb{N}$ be the set of natural numbers, $\{0, 1, 2, ...\}$. Let $\bot$ denote the divergent computation.[1] $\mathbb{N}_\bot \stackrel{\text{def}}{=} \mathbb{N} \cup \{\bot\}$. Let $\langle \cdot, \cdot \rangle$ be any fixed pairing function.[2] A function $f : \mathbb{N} \to \mathbb{N}_\bot$ is *effective* $\stackrel{\text{def}}{\Leftrightarrow}$ it is partial computable [Rog67]. A function $\psi : \mathbb{N} \to \mathcal{PC}$ is *effective* $\stackrel{\text{def}}{\Leftrightarrow}$ $\lambda p, x \centerdot \psi(p)(x)$ is partial computable.[3] For any set X, a function $\psi : \mathbb{N} \to X$ is a *numbering of* X [Rog58, Ric80, Roy87, Spr90, BGS03] $\stackrel{\text{def}}{\Leftrightarrow}$ ψ is onto.[4] We will often write ψ_p as shorthand for $\psi(p)$.

An effective numbering of type $\mathbb{N} \to \mathcal{PC}$ can be thought of as a *programming language*, in the following sense. If one were to take the programs in some programming language for $\mathcal{PC}$, and number those programs, e.g., length-lexicographically, then the function that sends p to the semantics of the pth program would be an effective numbering of type $\mathbb{N} \to \mathcal{PC}$.

The following are the formal statements of Kleene's Recursion Theorem (krt) and Rogers' Fixpoint Recursion Theorem (fprt).

Definition 1. For each effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$, (a) and (b) below.

(a) **(Kleene [Rog67, page 214, problem 11-4])** krt *holds in* $\psi \Leftrightarrow$

$$(\forall g \in \mathcal{PC})(\exists e)[\psi_e = g(\langle e, \cdot \rangle)]. \tag{1}$$

(b) **(Rogers [Rog67, Theorem 11-I])** fprt *holds in* $\psi \Leftrightarrow$

$$(\forall \text{ computable } t : \mathbb{N} \to \mathbb{N})(\exists e)[\psi_e = \psi_{t(e)}]. \tag{2}$$

krt can be interpreted as follows. The partial computable function g in (1) is an arbitrary, algorithmic task to perform with a self-copy; the program e is one that creates a copy of itself (external to itself) and then performs that task using this self-copy. In an important sense, this self-copy provides e *complete, low-level self-knowledge*. Thus, e can reflect upon its own *intensional* (synonym: *connotational*) characteristics, e.g., its size, runtime, memory usage, etc. Of course, e can *run* its self-copy, and thereby reflect upon its *extensional* (synonym: *denotational*) characteristics as well [Roy87].

fprt can be interpreted as follows. The function t in (2) is a transformation on programs; the program e is one whose semantics remain fixed under this transformation [Roy87].

The following constructive variant of fprt is also sometimes considered.[5]

Definition 2 (Riccardi [Ric80, Definition 2.1]). For each effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$, FPRT *holds in* $\psi \Leftrightarrow$ there exists a computable function $n : \mathbb{N} \to \mathbb{N}$ such that, for each p,

$$\begin{aligned} &[(\psi_p \circ n)(p) \neq \bot \;\Rightarrow\; \psi_{n(p)} = \psi_{(\psi_p \circ n)(p)}] \\ \wedge\; &[(\psi_p \circ n)(p) = \bot \;\Rightarrow\; \psi_{n(p)} = \lambda x \centerdot \bot]. \end{aligned} \tag{3}$$

[1] Thus, $\bot$ may be thought of as the *value* of an *infinite loop*.
[2] A *pairing function* is computable, 1-1, onto, and of type $\mathbb{N}^2 \to \mathbb{N}$ [Rog67, page 64].
[3] In Section 2, the notion of *effective* is generalized to other types of functions.
[4] In this paper, we shall generally use lowercase Greek letters (e.g., γ, ψ) for numberings, and lowercase Roman letters (e.g., f, g, h) for other functions.
[5] krt similarly has constructive variants, which are considered, e.g., in [Roy87, CM09a].

```
#include <stdio.h>
int main(){char*s[]={"#include <stdio.h>%cint main(){char*s[]={",
"};printf(s[0],10);int i=0;while(i<3)printf(%c%%c%%s%%c,%%c%c,34,%c",
"s[i++],34,10);printf(s[1],34,34,10);printf(s[2],10);return 0;}%c",
};printf(s[0],10);int i=0;while(i<3)printf("%c%s%c,%c",34,
s[i++],34,10);printf(s[1],34,34,10);printf(s[2],10);return 0;}
```

$$\lambda e.\, e$$

```
#include <stdio.h>
int main() { printf("75");
             return 0;     }
```

$$\lambda e.\, \lceil \log_{256}(e+1) \rceil$$

```
#include <stdio.h>
int main() { printf("#");
             return 0;    }
```

$$\lambda e.\, (e \bmod 256)$$

Fig. 1. Top: A C-program that outputs its own source code. **Bottom-Left (BL)**: A C-program that outputs the *length* of its source code (in bytes). **Bottom-Right (BR)**: A C-program that outputs the *first character* of its source code. Also depicted are functions of type $\mathbb{N} \to \mathbb{N}_\perp$ corresponding to each C-program.

In (3), ψ_p and $n(p)$ play the roles played by t and e, respectively, in (2). In this sense, the function n *finds* a program $n(p)$ whose semantics remain fixed under the transformation ψ_p. Such an n is called an *effective instance of* FPRT *in* ψ.

fprt was popularized by Rogers' classic textbook [Rog67]. Therein, he writes [Rog67, page 182]: "Kleene's formulation differs slightly, and inessentially, from ours." It is true that, in any *standard* numbering of $\mathcal{PC}$, both theorems will hold. More broadly, in any effective numbering (standard or otherwise) in which krt holds, fprt holds as well [Ric80, Theorem 5.1]. However, there *do* exist non-standard, effective numberings of $\mathcal{PC}$ in which fprt holds, but in which krt does *not* hold [Ric80, Theorem 5.3].[6] In this sense, krt is strictly *stronger* than fprt.

Given that krt is stronger in this sense, are there programs (for $\mathcal{PC}$) that one would reasonably call self-referential, and whose existence is assured by krt, but *not* by fprt?

Consider the three C-programs in Figure 1. Figure 1(Top) is a program that outputs its own source code (such a *self-reproducing* program is sometimes called a *quine* [Tho99]);[7] Figure 1(BL) is a program that outputs the *length* of its source code (in bytes); and Figure 1(BR) is a program that outputs the *first character* of its source code.

Most would surely agree that a self-reproducing program, like that of Figure 1(Top), is self-referential. For the programs in Figures 1(BL) and 1(BR), how-

[6] The effective numbering of $\mathcal{PC}$ used in the proof of [Ric80, Theorem 5.3] is the same as that used in the proof of [MWY78, Theorem 3.6].

[7] Early examples of such programs are due to Lee [Lee63] and Thatcher [Tha63]. The term "quine" appears to have been inspired by Hofstadter [Hof79]. Therein, Hofstadter refers to the operation of *preceding a phrase by its own quotation* as "quining", in honor of Willard Van Orman Quine [Qui66].

ever, it is less clear. For example, many C-programs begin with the character '#'. Thus, one could argue that the program in Figure 1(BR) is more *opportunistic* than self-referential.

Common among these three C-programs is that each computes a function of the form $\lambda x \boldsymbol{.} f(e)$, for some effective function $f : \mathbb{N} \to \mathbb{N}_\perp$; that is, each outputs the result of applying a (partial) computable function to its source code e, whilst ignoring any possible input x.[8] These functions are also depicted in Figure 1.[9] For example, for the self-reproducing program in Figure 1(Top), the corresponding function f is simply $\lambda e \boldsymbol{.} e$.[10]

Clearly, for each effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$ in which krt holds, and for each effective $f : \mathbb{N} \to \mathbb{N}_\perp$, there exists an e such that

$$\psi_e = \lambda x \boldsymbol{.} f(e). \tag{4}$$

But what if merely fprt holds in ψ? Then, can the same still be said *for each f*?

The answer is "no". Our first main result, Theorem 3, says that fprt assures the existence of a program e as in (4) for precisely those effective $f : \mathbb{N} \to \mathbb{N}_\perp$ with finite range.[11]

Theorem 3. Suppose that $f : \mathbb{N} \to \mathbb{N}_\perp$ is effective. Then, (a)-(c) below are equivalent.

(a) For every effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$ in which fprt holds, there exists an e such that $\psi_e = \lambda x \boldsymbol{.} f(e)$.
(b) For every effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$ in which FPRT holds, there exists an e such that $\psi_e = \lambda x \boldsymbol{.} f(e)$.
(c) The range of f is finite.

Thus, fprt does *not* assure the existence of a program like that of Figure 1(Top), *nor* even like that of Figure 1(BL) (though fprt does assure the existence of a program like that of Figure 1(BR)). Given that most would call the program of Figure 1(Top) self-referential, Theorem 3 suggests that krt is better than fprt at capturing the notion of program self-reference *in* $\mathcal{PC}$.

One can consider self-reproducing programs in effective numberings other than those of type $\mathbb{N} \to \mathcal{PC}$. For example, consider an effective numbering $\psi : \mathbb{N} \to \mathbb{N}_\perp$. (Thus, programs in this ψ code elements of $\mathbb{N}_\perp$, instead of $\mathcal{PC}$.) In such a numbering, one could reasonably call a program e *self-reproducing* $\Leftrightarrow$

[8] In the terminology of [Cas71], $f(e)$ is the computable distortion of e wrought by f.

[9] ISO-8859-1 is a character encoding commonly used on the Internet (see, for example, [Moo96]). This encoding associates a character with each value in $\{0, ..., 255\}$. One can treat each program in Figure 1 as a base-256 number, e.g., where each character is a digit whose value is determined by the ISO-8859-1 encoding, and where the leading character is *least* significant. In this way, "`#include ...`" mod 256 = "`#`".

[10] Thus, $\psi_e = \lambda x \boldsymbol{.} f(e) = \lambda x \boldsymbol{.} (\lambda e \boldsymbol{.} e)(e) = \lambda x \boldsymbol{.} e$.

[11] The proof of Theorem 3 employs, among other things, techniques of Machtey, Winklmann, and Young [MWY78, proof of Theorem 3.6]. The proofs of Theorem 3 and of all subsequent results appear in the appendix.

$\psi(e) = e$. Then, one could ask the following. Does fprt assure the existence of a self-reproducing program in an effective numbering *of this type*? More generally, is fprt still *weaker* than krt in an effective numbering *of this type*?

To state such questions formally, one must first make precise what one *means* by krt and fprt in such numberings. To do so, we first introduce the notion of a *constructive Scott subdomain*. Then, in Section 3 we generalize krt to effective numberings of arbitrary constructive Scott subdomains. (For fprt, a similar generalization was already known [Ers77].) Finally, in Section 4, we revisit the aforementioned questions.

2 Constructive Scott Subdomains

In this section, we introduce *constructive Scott subdomains*. Intuitively, a constructive Scott subdomain is a collection of objects with the property that: each object can be broken up into *pieces*. $\mathcal{PC}$ is an example of such a collection, in which case the pieces are the finite functions over $\mathbb{N}$.

Having such a collection can be useful, as it can often be easier to work with each object *in pieces*, than to work with each object *as a whole*. For example, consider the effective numberings of type $\mathbb{N} \to \mathcal{PC}$. In such a numbering ψ, the predicate $\lambda p, q\,.\,[\psi_p \subseteq \psi_q]$ is *never* computable [JST09, Proposition 14]. However, if $\lambda i\,.\,F_i$ is a canonical enumeration of the finite functions over $\mathbb{N}$ [Rog67, MY78], then the predicate $\lambda i, j\,.\,[F_i \subseteq F_j]$ *is* computable.

As the reader will be able to see following the formal definitions, many collections of objects can be viewed in this way. Unless otherwise noted, concepts not explained below can be found in [SHLG94].

As is customary, we use $\sqsubseteq$ to denote the ordering relation of any given partial order. A partial order X is *flat* $\overset{\text{def}}{\Leftrightarrow}$ X has a least element $\bot$, and X is ordered such that, for each $x, y \in X$, $x \sqsubseteq y \;\Leftrightarrow\; [x = \bot \,\vee\, x = y]$.[12] Such is the standard ordering of $\mathbb{N}_\bot$. Thus, $\mathbb{N}_\bot$ is an example of a flat partial order.

For a subset A of a partial order X, $\bigsqcup A$ denotes the least-upper-bound of A, which may or may not *exist*, or may or may not exist *in* X. For the special case of a two element set: $x \sqcup y \overset{\text{def}}{=} \bigsqcup\{x, y\}$. A subset A of a partial order is *directed* $\overset{\text{def}}{\Leftrightarrow}$ A is *non*-empty and, for each $x, y \in A$, $\{x, y\}$ has some (not necessarily least) upper-bound in A. For each partial order X, and each $x \in X$, x is *compact* in X $\overset{\text{def}}{\Leftrightarrow}$ $(\forall \text{ directed } A \subseteq X)\big[[\bigsqcup A \text{ exists in } X \;\wedge\; x \sqsubseteq \bigsqcup A] \;\Rightarrow\; (\exists y \in A)[x \sqsubseteq y]\big]$. For each partial order X, $\mathsf{K}(X) \overset{\text{def}}{=} \{x \in X \mid x \text{ is compact in } X\}$. For example, $\mathsf{K}(\mathbb{N}_\bot) = \mathbb{N}_\bot$. Similarly, if one lets $\mathcal{P}$ be the set of all functions of type $\mathbb{N} \to \mathbb{N}_\bot$ ordered pointwise (i.e., for each $f, g \in \mathcal{P}$, $f \sqsubseteq g \;\Leftrightarrow\; (\forall x)[f(x) \sqsubseteq g(x)]$), then $\mathsf{K}(\mathcal{P})$ is exactly the set of finite functions over $\mathbb{N}$.

A *Scott domain* is a partial order D satisfying (a)-(d) below.

(a) D has a least element (denoted by $\bot$).
(b) D is *complete*: for every directed $A \subseteq D$, $\bigsqcup A$ exists in D.

[12] An anonymous referee attributes this notion to Scott. Flat partial orders are also considered in [SHLG94, Spr98].

(c) D is *algebraic*: for each $y \in D$, the set $A = \{x \in \mathsf{K}(D) \mid x \sqsubseteq y\}$ is directed and $y = \bigsqcup A$.
(d) D is a *conditional upper semi-lattice*: for each $x, y \in D$, if $\{x, y\}$ has an upper-bound in D, then $x \sqcup y$ exists in D.

Suppose that D is a Scott domain, γ is a numbering of $\mathsf{K}(D)$, and S is such that $\mathsf{K}(D) \subseteq S \subseteq D$. A function $\psi : \mathbb{N} \to S$ is *effective via* $\gamma \overset{\text{def}}{\Leftrightarrow}$ the set $\{\langle i, p\rangle \mid \gamma(i) \sqsubseteq \psi(p)\}$ is computably enumerable (ce) [Rog67].[13] Recall that Section 1 defined *effective* for functions of type $\mathbb{N} \to \mathbb{N}_\bot$ and $\mathbb{N} \to \mathcal{PC}$. In the former case, the definition is equivalent to *effective via* $\text{pred} : \mathbb{N} \to \mathsf{K}(\mathbb{N}_\bot)$, where, for each x,

$$\text{pred}(x) = \begin{cases} \bot, & \text{if } x = 0; \\ x - 1, & \text{otherwise.} \end{cases} \tag{5}$$

In the latter case, the definition is equivalent to *effective via* $\lambda i \centerdot F_i : \mathbb{N} \to \mathsf{K}(\mathcal{P})$.[14]

A *constructive Scott subdomain* [SHLG94, Spr98, Spr07] is a tuple (D, γ, S) satisfying (a)-(g) below.

(a) D is a Scott domain.
(b) γ is a numbering of $\mathsf{K}(D)$.
(c) The predicate $\lambda i, j \centerdot [\gamma(i) \sqsubseteq \gamma(j)]$ is computable.
(d) The predicate $\lambda i, j \centerdot [\{\gamma(i), \gamma(j)\}$ has an upper-bound in $D]$ is computable.
(e) There exists a computable function $u : \mathbb{N}^2 \to \mathbb{N}$ such that, if $\{\gamma(i), \gamma(j)\}$ has an upper-bound in D, then $(\gamma \circ u)(i, j) = \gamma(i) \sqcup \gamma(j)$.
(f) $\mathsf{K}(D) \subseteq S \subseteq D$.
(g) There exists an effective numbering $\psi : \mathbb{N} \to S$ via γ.

We shall sometimes refer to a ψ as in (g) just above as an *effective numbering of* (D, γ, S).

A constructive Scott subdomain (D, γ, S) is *recursively complete* [SHLG94, page 270, Definition 4.8] $\overset{\text{def}}{\Leftrightarrow}$ $(\forall\ \mathsf{ce}\ A \subseteq \mathbb{N})[\gamma(A)$ is directed $\Rightarrow$ $\bigsqcup \gamma(A) \in S]$. For example, of the two constructive Scott subdomains $(\mathcal{P}, \lambda i \centerdot F_i, \mathcal{PC})$ and $(\mathcal{P}, \lambda i \centerdot F_i, \mathsf{K}(\mathcal{P}))$, the former is recursively complete, while the latter is *not*.

3 krt in Constructive Scott Subdomains

In this section, we generalize krt to arbitrary constructive Scott subdomains. In order to give some intuition for the generalized definition, let us first return attention to effective numberings of type $\mathbb{N} \to \mathcal{PC}$. Note that, for such numberings,

[13] A function $\psi : \mathbb{N} \to S$ can be effective via γ, but *not* via γ'. The following is an example. Let K be the diagonal halting problem [Rog67, page 62]. Let $\{k_0 < k_1 < \cdots\} = K$ and $\{\bar{k}_0 < \bar{k}_1 < \cdots\} = \bar{K} \overset{\text{def}}{=} \mathbb{N} - K$. Let $f : \mathbb{N} \to \mathbb{N}$ be such that, for each i, $f(2i) = k_i$ and $f(2i + 1) = \bar{k}_i$. Let $\wp(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$ ordered by $\subseteq$, and let $\lambda i \centerdot D_i : \mathbb{N} \to \mathsf{K}(\wp(\mathbb{N}))$ be a canonical enumeration of the finite subsets of $\mathbb{N}$ [Rog67, MY78]. Note that, for each i, $f(D_i) \subseteq \bar{K} \Leftrightarrow D_i \subseteq 2\mathbb{N} + 1$. Thus, $\lambda p \centerdot \bar{K} : \mathbb{N} \to \{\bar{K}\}$ is effective via $\lambda i \centerdot f(D_i)$. However, $\lambda p \centerdot \bar{K}$ is *not* effective via $\lambda i \centerdot D_i$, as this would imply $\{i \mid D_i \subseteq \bar{K}\}$ is computably enumerable.

[14] In neither case is the choice of γ *unique*.

the following definition of krt is *equivalent* to Definition 1(a). For each effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$, krt holds in $\psi \Leftrightarrow$

$$(\forall \text{ effective } f : \mathbb{N} \to \mathcal{PC})(\exists e)[\psi_e = f(e)]. \tag{6}$$

To see that this is equivalent, note that, for each $g \in \mathcal{PC}$, there exists an effective $f : \mathbb{N} \to \mathcal{PC}$ such that $(\forall p, x)[f(p)(x) = g(\langle p, x\rangle)]$, and vice versa.

Taking a cue from (6), we define krt for arbitrary constructive Scott subdomains as follows.

Definition 4. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, for each effective numbering $\psi : \mathbb{N} \to S$ via γ, krt *holds in* $\psi \Leftrightarrow$

$$(\forall \text{ effective } f : \mathbb{N} \to S \text{ via } \gamma)(\exists e)[\psi(e) = f(e)]. \tag{7}$$

This definition preserves many of the intuitive properties held by krt in effective numberings of type $\mathbb{N} \to \mathcal{PC}$. For example, as will be seen in Section 4, the fact that krt *entails* fprt in such numberings carries over to effective numberings of arbitrary constructive Scott subdomains.

Suppose that (D, γ, S) is a constructive Scott subdomain. Natural questions to ask are the following. Do there exist effective numberings of (D, γ, S) in which krt holds? Do there exist such numberings in which krt does *not* hold? Our next main result, Theorem 5, gives conditions for the existence of each kind of numbering.

Theorem 5. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, (a) and (b) below.

(a) There exists an effective numbering $\psi : \mathbb{N} \to S$ via γ in which krt holds $\Leftrightarrow$ (D, γ, S) is recursively complete.
(b) There exists an effective numbering $\psi : \mathbb{N} \to S$ via γ in which krt does *not* hold $\Leftrightarrow D \neq \{\bot\} \Leftrightarrow S \neq \{\bot\}$.

Definition 6 just below introduces the notion of a *regular* partial order.

Definition 6 (Spreen [Spr00]). A partial order D is *regular* $\Leftrightarrow$ for each $x, y \in \mathsf{K}(D)$,

$$y \not\sqsubseteq x \;\Rightarrow\; \big(\exists z \in \mathsf{K}(D)\big)[x \sqsubseteq z \;\wedge\; \{y, z\} \text{ has } \textit{no} \text{ upper-bound in } \mathsf{K}(D)].^{15} \tag{8}$$

An example of a regular partial order is $\mathcal{P}$.[16] To see this, let f and g be finite functions over $\mathbb{N}$ such that $g \not\subseteq f$. If $f \cup g$ is not single valued, then already

[15] We shall generally be interested in regular *Scott domains*. It can be shown that, for each Scott domain D, and each $x, y \in \mathsf{K}(D)$, $\{x, y\}$ has an upper-bound in $\mathsf{K}(D) \Leftrightarrow \{x, y\}$ has an upper-bound in D. (See, for example, [SHLG94, page 57, Lemma 1.9].) Thus, if D is a Scott domain, then D is regular $\Leftrightarrow$ for each $x, y \in \mathsf{K}(D)$,

$$y \not\sqsubseteq x \;\Rightarrow\; \big(\exists z \in \mathsf{K}(D)\big)[x \sqsubseteq z \;\wedge\; \{y, z\} \text{ has } \textit{no} \text{ upper-bound in } D].$$

[16] Recall that $\mathcal{P}$ is the set of all functions of type $\mathbb{N} \to \mathbb{N}_\bot$ ordered pointwise.

$\{f, g\}$ has no upper-bound in $\mathcal{P}$. On the other hand, if there exists an $x_0 \in \mathbb{N}$ such that $f(x_0) = \bot \neq g(x_0)$, then $f \subseteq h$ and $\{g, h\}$ has no upper-bound in $\mathcal{P}$, where, for each x,

$$h(x) = \begin{cases} g(x_0) + 1, & \text{if } x = x_0; \\ f(x), & \text{otherwise.} \end{cases} \tag{9}$$

An example of a *non*-regular partial order is $\wp(\mathbb{N})$, the set of all subsets of $\mathbb{N}$ ordered by $\subseteq$.

Our next main result, Theorem 7, *characterizes* the effective numberings in which krt holds, for the constructive Scott subdomains (D, γ, S) for which D is regular. Note that, in the definition of krt for arbitrary constructive Scott subdomains (Definition 4), one could write (7) as

$$(\forall \text{ effective } f : \mathbb{N} \to S \text{ via } \gamma)(\exists e)[\psi(e) \sqsubseteq f(e) \ \wedge \ f(e) \sqsubseteq \psi(e)]. \tag{10}$$

Theorem 7 says that, if (D, γ, S) is a constructive Scott subdomain and D is regular, then having just the latter half of (10) suffices to have full krt.[17]

Theorem 7. Suppose that (D, γ, S) is a constructive Scott subdomain and that D is regular. Then, for each effective numbering $\psi : \mathbb{N} \to S$ via γ, (a) and (b) below are equivalent.

(a) krt holds in ψ.
(b) $(\forall \text{ effective } f : \mathbb{N} \to S \text{ via } \gamma)(\exists e)[f(e) \sqsubseteq \psi(e)]$.

$\mathcal{P}$ is an example of a regular Scott domain. Thus, Theorem 7's characterization holds in the effective numberings of $(\mathcal{P}, \lambda i \centerdot F_i, \mathcal{PC})$. (In some sense, $(\mathcal{P}, \lambda i \centerdot F_i, \mathcal{PC})$ represents the *standard* way of formulating $\mathcal{PC}$ as a constructive Scott subdomain.)

Our next main result, Theorem 8, says that, if (D, γ, S) is a constructive Scott subdomain and D is *not* regular, then there exists an effective numbering of (D, γ, S) in which Theorem 7's characterization *fails*. Theorem 7 and Theorem 8 together say that Theorem 7's characterization holds in the effective numberings of a constructive Scott subdomain (D, γ, S) iff D is regular.

Theorem 8. Suppose that (D, γ, S) is a constructive Scott subdomain and that D is *not* regular. Then, there exists an effective numbering $\psi : \mathbb{N} \to S$ via γ satisfying (a) and (b) below.

(a) $(\forall \text{ effective } f : \mathbb{N} \to S \text{ via } \gamma)(\exists e)[f(e) \sqsubseteq \psi(e)]$.
(b) krt does *not* hold in ψ.

$\wp(\mathbb{N})$ is an example of a *non*-regular Scott domain. Thus, if one lets $\lambda i \centerdot D_i : \mathbb{N} \to \mathsf{K}(\wp(\mathbb{N}))$ be a canonical enumeration of the finite subsets of $\mathbb{N}$ [Rog67, MY78], and if one lets $\mathcal{CE}$ be the collection of all computably enumerable sets, then Theorem 7's characterization fails in the effective numberings of $(\wp(\mathbb{N}), \lambda i \centerdot D_i, \mathcal{CE})$. (In some sense, $(\wp(\mathbb{N}), \lambda i \centerdot D_i, \mathcal{CE})$ represents the *standard* way of formulating $\mathcal{CE}$ as a constructive Scott subdomain.)

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[17] The proof of Theorem 7 bears some resemblance to Royer's proof of [Roy87, Theorem 4.2.15].

4 krt and fprt in Constructive Scott Subdomains

In this section, we compare krt and fprt in arbitrary constructive Scott subdomains. The generalization of fprt to arbitrary constructive Scott subdomains is essentially due to Ershov.[18]

Definition 9 (Ershov [Ers77]). Suppose that (D, γ, S) is a constructive Scott subdomain. Then, for each effective numbering $\psi : \mathbb{N} \to S$ via γ, fprt *holds in* $\psi \Leftrightarrow$

$$(\forall \text{ computable } t : \mathbb{N} \to \mathbb{N})(\exists e)[\psi(e) = (\psi \circ t)(e)]. \tag{11}$$

Recall from Section 1 that krt *entails* fprt in effective numberings of type $\mathbb{N} \to \mathcal{PC}$. Does this entailment relationship carry over to effective numberings of arbitrary constructive Scott subdomains? Proposition 10 just below says that the answer is "yes".[19]

Proposition 10. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, for each effective numbering $\psi : \mathbb{N} \to S$ via γ, if krt holds in ψ, then fprt holds in ψ as well.

Also recall from Section 1 that there exist effective numberings of $\mathcal{PC}$ in which fprt holds, but in which krt does *not* hold [Ric80, Theorem 5.3]. Do there exist such numberings for arbitrary constructive Scott subdomains?

The answer, as it turns out, is that such numberings exist for *some* constructive Scott subdomains, but *not* for others. Note that, for subdomains of the latter kind, krt and fprt are *equivalent*, by Proposition 10. Our final main result, Theorem 11, says that, for a constructive Scott subdomain (D, γ, S), this equivalence occurs precisely when D is flat. In such a subdomain, krt and fprt capture the notion of program self-reference equally well.

Theorem 11. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, (a) and (b) below are equivalent.

(a) $(\forall \text{ effective } \psi : \mathbb{N} \to S \text{ via } \gamma)[$krt holds in $\psi \Leftrightarrow$ fprt holds in $\psi]$.
(b) D is flat.

$\mathbb{N}_\perp$ is an example of a flat Scott domain.[20] Thus, this equivalence occurs in any constructive subdomain of $\mathbb{N}_\perp$. Recall from Section 3 that, in an effective numbering $\psi : \mathbb{N} \to \mathbb{N}_\perp$, one could reasonably call a program e *self-reproducing* $\Leftrightarrow \psi(e) = e$. Clearly, in such a numbering, krt suffices to assure the existence of such a program. Furthermore, by Theorem 11, fprt is equally sufficient.

[18] See, for example, [BGS03, Theorem 2.1].
[19] Proposition 10 generalizes Riccardi's [Ric80, Theorem 5.1].
[20] Of course, $\mathcal{PC}$ and $\mathcal{CE}$ are *not* flat. Thus, neither are $\mathcal{P}$ nor $\wp(\mathbb{N})$.

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Appendix

For each $f : X \to Y$, $\mathrm{rng}(f) \stackrel{\mathrm{def}}{=} \{y \in Y \mid (\exists x \in X)[f(x) = y]\}$. Note that, if $f : \mathbb{N} \to \mathbb{N}_\perp$ and $(\exists x)[f(x) = \perp]$, then $\perp \in \mathrm{rng}(f)$, which some might consider unconventional. For such an f, should we wish to work with $\mathrm{rng}(f) - \{\perp\}$, we shall say so explicitly.

An effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$ is *acceptable* $\stackrel{\mathrm{def}}{\Leftrightarrow}$ for every effective numbering $\xi : \mathbb{N} \to \mathcal{PC}$, there exists a computable function $t : \mathbb{N} \to \mathbb{N}$ such that $(\forall p)[\psi_{t(p)} = \xi_p]$ [Rog58, Rog67, MWY78, MY78, Ric80, Roy87]. Let $\varphi : \mathbb{N} \to \mathcal{PC}$ be a fixed acceptable effective numbering. For each p, let $W_p = \{x \mid \varphi_p(x) \neq \perp\}$.

Lemma 1. Suppose that $h : \mathbb{N} \to \mathbb{N}$ is computable, and that $\mathrm{rng}(h)$ is infinite. Then, there exist effective numberings $\theta, \xi : \mathbb{N} \to \mathcal{PC}$ satisfying (a) and (b) below.

(a) $(\forall i, p)\big[[\mathrm{rng}(\theta_i) \subseteq \{\perp, h(p)\} \ \wedge\ \xi_p(0) = i] \ \Rightarrow\ \theta_i \text{ is finite}\big]$.
(b) ξ is acceptable.

Proof. Let h be as stated. For each q and x, and each $j \in \{0, 1\}$, let θ and ξ be as follows.

$$\theta_{2q+j}(x) = \begin{cases} \varphi_q(x), & \text{if } (\forall p < x)[\xi_p^x(0) = 2q + j \ \Rightarrow\ \mathrm{rng}(\varphi_q) \not\subseteq \{\perp, h(p)\}]; \\ \perp, & \text{otherwise.} \end{cases} \tag{12}$$

$$\xi_q(x) = \begin{cases} \varphi_n(x), & \text{if } (\forall p < q)[h(p) \neq h(q)], \text{ where } n = |\{h(p) \mid p < q\}|; \\ \perp, & \text{otherwise.} \end{cases} \tag{13}$$

Claim 1.1. ξ is an acceptable numbering of $\mathcal{PC}$.

Proof of Claim. Let t be such that, for each n,

$$t(n) = q, \text{ where } q \text{ is } \textit{least} \text{ such that } n = |\{h(p) \mid p < q\}|. \tag{14}$$

Clearly, t is partial computable. Furthermore, since $\mathrm{rng}(h)$ is infinite, t is total. Clearly, for each n, $\xi_{t(n)} = \varphi_n$. □ (**Claim 1.1**)

Claim 1.2. For each p, if $\xi_p \neq \lambda x.\perp$, then, for each $q > p$ such that $h(q) = h(p)$, $\xi_q = \lambda x.\perp$.

Proof of Claim. Clear by the construction of ξ. □ (**Claim 1.2**)

Claim 1.3. θ is an effective numbering of $\mathcal{PC}$.

Proof of Claim. To show the claim, it suffices to show that, for each q, there exists an i such that $\theta_i = \varphi_q$. Let q be fixed. The only interesting case is when there exists a p such that

$$\mathrm{rng}(\varphi_q) \subseteq \{\perp, h(p)\} \ \wedge\ \xi_p(0) \in \{2q, 2q + 1\}. \tag{15}$$

By Claim 1.2, there can exist *at most one* such p. Suppose that $\xi_p(0) = 2q + j$, where $j \in \{0, 1\}$. Then, clearly, by the construction of θ, $\theta_{2q+(1-j)} = \varphi_q$.
□ (**Claim 1.3**)

Claim 1.4. $(\forall i, p)\big[[\mathrm{rng}(\theta_i) \subseteq \{\bot, h(p)\} \ \wedge\ \xi_p(0) = i] \ \Rightarrow\ \theta_i \text{ is finite}\big]$.

Proof of Claim. Let i and p be such that

$$\mathrm{rng}(\theta_i) \subseteq \{\bot, h(p)\} \ \wedge\ \xi_p(0) = i. \tag{16}$$

Let x_0 be such that $\xi_p^{x_0}(0) = i$. Clearly, for each $x \geq x_0$, $\theta_i(x) = \bot$.
□ (**Claim 1.4**)
□ (**Lemma 1**)

Theorem 3. Suppose that $f : \mathbb{N} \to \mathbb{N}_\bot$ is effective. Then, (a)-(c) below are equivalent.

(a) For every effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$ in which fprt holds, there exists an e such that $\psi_e = \lambda x \centerdot f(e)$.
(b) For every effective numbering $\psi : \mathbb{N} \to \mathcal{PC}$ in which FPRT holds, there exists an e such that $\psi_e = \lambda x \centerdot f(e)$.
(c) $\mathrm{rng}(f)$ is finite.

Proof. (a) $\Rightarrow$ (b): Immediate.

(b) $\Rightarrow$ (c): (By contrapositive.) Suppose that $\mathrm{rng}(f)$ is *infinite*. Let $g : \mathbb{N} \to \mathbb{N}$ be a monotone increasing computable function such that, for each p,

$$(f \circ g)(p) \notin \big(\{\bot\} \cup \{g(0), ..., g(p-1)\}\big). \tag{17}$$

Since $\mathrm{rng}(f)$ is infinite, such a g clearly exists. Note that $f \circ g$ is computable, and $\mathrm{rng}(f \circ g)$ is infinite. Thus, there exist effective numberings $\theta, \xi : \mathbb{N} \to \mathcal{PC}$ as in Lemma 1 for $h = f \circ g$. Let i_0 be such that $\theta_{i_0} = \lambda x \centerdot x$. For each q, let ξ' be as follows.

$$\xi'_q = \begin{cases} \lambda x \centerdot i_0, & \text{if } q \notin \mathrm{rng}(g); \\ \xi_{g^{-1}(q)}, & \text{otherwise.} \end{cases} \tag{18}$$

Clearly, ξ' is an *acceptable* numbering of $\mathcal{PC}$. For each p, let ψ be as follows.

$$\psi_p = \begin{cases} \theta_{\xi'_p(0)}, & \text{if } \xi'_p(0) \neq \bot; \\ \lambda x \centerdot \bot, & \text{otherwise.} \end{cases} \tag{19}$$

Claim 3.1. ψ is an effective numbering of $\mathcal{PC}$.

Proof of Claim. To show the claim, it suffices to show that, for each i, there exists a p such that $\psi_p = \theta_i$. Let i be fixed, and let p be such that $\xi'_p = \lambda x \centerdot i$. Then, $\psi_p = \theta_{\xi'_p(0)} = \theta_{(\lambda x \centerdot i)(0)} = \theta_i$. □ (**Claim 3.1**)

Claim 3.2. For each p and q, if $\xi'_p = \xi'_q$, then $\psi_p = \psi_q$.

Proof of Claim. Easily verifiable from (19). □ (**Claim 3.2**)

Claim 3.3. FPRT holds in ψ.

Proof of Claim. Since ξ' is an acceptable numbering of $\mathcal{PC}$, there exist computable functions $n, t : \mathbb{N} \to \mathbb{N}$ such that

- n witnesses FPRT in ξ', and
- for each p, $\xi'_{t(p)} = \psi_p$.

The remainder of the proof of the claim is to show that $n \circ t$ witnesses FPRT in ψ. Let ψ-program p be fixed, and consider the following cases.

Case $(\xi'_{t(p)} \circ n \circ t)(p) \neq \bot$. Then,

$$\xi'_{(n \circ t)(p)} = \xi'_{(\xi'_{t(p)} \circ n \circ t)(p)}. \tag{20}$$

Furthermore, by the case and the choice of t,

$$(\psi_p \circ n \circ t)(p) \neq \bot. \tag{21}$$

Thus,

$$\begin{aligned} \psi_{(n \circ t)(p)} &= \psi_{(\xi'_{t(p)} \circ n \circ t)(p)} && \{\text{by Claim 3.2 and (20)}\} \\ &= \psi_{(\psi_p \circ n \circ t)(p)} && \{\text{by the choice of } t\}. \end{aligned}$$

Case $(\xi'_{t(p)} \circ n \circ t)(p) = \bot$. Then,

$$\xi'_{(n \circ t)(p)} = \lambda x \centerdot \bot. \tag{22}$$

Furthermore, by the case and the choice of t,

$$(\psi_p \circ n \circ t)(p) = \bot. \tag{23}$$

By (22), $\xi'_{(n \circ t)(p)}(0) = \bot$. Thus, by (19), $\psi_{(n \circ t)(p)} = \lambda x \centerdot \bot$. □ (**Claim 3.3**)

Claim 3.4. $(\forall q)[\psi_q \neq \lambda x \centerdot f(q)]$.

Proof of Claim. By way of contradiction, let q be such that $\psi_q = \lambda x \centerdot f(q)$. Consider the following cases.

Case $q \notin \mathrm{rng}(g)$. Then, $\xi'_q = \lambda x \centerdot i_0$. (Recall: $\theta_{i_0} = \lambda x \centerdot x$.) Thus,

$$\psi_q = \theta_{\xi'_q(0)} = \theta_{(\lambda x \centerdot i_0)(0)} = \theta_{i_0} = \lambda x \centerdot x \tag{24}$$

— a contradiction.

Case $q \in \mathrm{rng}(g)$. Let $p = g^{-1}(q)$. Thus, $\psi_q = \lambda x \centerdot (f \circ g)(p)$ and $\xi'_q = \xi_p$. By the case, $f(q) \neq \bot$, and, thus, $\psi_q \neq \lambda x \centerdot \bot$. It follows that, $\xi_p(0) = \xi'_q(0) \neq \bot$. Let $i = \xi_p(0)$. (Thus, $\psi_q = \theta_i$.) Then,

$$\mathrm{rng}(\theta_i) \subseteq \{\bot, (f \circ g)(p)\} \ \wedge\ \xi_p(0) = i \ \wedge\ \{x \mid \theta_i(x) \neq \bot\} \text{ is infinite.} \tag{25}$$

But this contradicts (a) of Lemma 1 for θ and ξ. □ (**Claim 3.4**)

(c) $\Rightarrow$ (a): By way of contradiction, let $f : \mathbb{N} \to \mathbb{N}_\perp$ be an effective function witnessing otherwise. Let ψ be any effective numbering of $\mathcal{PC}$ in which fprt holds, and such that $(\forall p)[\psi_p \neq \lambda x \centerdot f(p)]$. Clearly,

$$(\forall p)[\psi_p = \lambda x \centerdot \perp \ \Rightarrow\ f(p) \neq \perp]. \tag{26}$$

Let $\{y_0 < y_1 < \cdots < y_{n-1}\} = \text{rng}(f) - \{\perp\}$. Let $y_n, y_{n+1} \in \mathbb{N}$ be any such that $|\{y_0, ..., y_{n+1}\}| = n + 2$. For each $i \leq n + 1$, let q_i be such that $\psi_{q_i} = \lambda x \centerdot y_i$. Let $t : \mathbb{N} \to \mathbb{N}$ be such that, for each p, $t(p)$ is determined as follows.

Dovetail between the following two steps until one applies. (By (26), one must eventually apply.)

(i) If $f(p) \neq \perp$, output q_i, where $y_i = f(p)$.
(ii) If there exists a $y \in \text{rng}(\psi_p) - \{\perp\}$, output y_i, where i is any such that $y_i \neq y$. (Note that such an i necessarily exists since $|\{y_0, ..., y_{n+1}\}| \geq 2$.)

Clearly, t is computable. By fprt in ψ, there exists an e such that $\psi_e = \psi_{t(e)}$. Consider the following cases.

Case [step (i) applies first in the computation of $t(e)$]. Let $y_i = f(e)$. Then, $\psi_e = \psi_{t(e)} = \psi_{q_i} = \lambda x \centerdot y_i = \lambda x \centerdot f(e)$ — a contradiction.

Case [step (ii) applies first in the computation of $t(e)$]. Let y be that which is discovered in $\text{rng}(\psi_e) - \{\perp\}$, and let i be such that $q_i = t(e)$. Thus, $y_i \neq y$. Furthermore, since $y \in \text{rng}(\psi_e)$, clearly, $\psi_e \neq \lambda x \centerdot y_i = \psi_{q_i} = \psi_{t(e)}$ — a contradiction. □ (**Theorem 3**)

Proposition 4. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, for each $f : \mathbb{N} \to S$, f is effective via $\gamma \Leftrightarrow$ there exists a computable function $\lambda s, p \centerdot f^s(p) : \mathbb{N}^2 \to \mathbb{N}$ such that, for each p,

$$f(p) = \bigsqcup\{(\gamma \circ f^s)(p) \mid s \in \mathbb{N}\}. \tag{27}$$

Proof of Proposition. Straightforward. □ (**Proposition 4**)

Theorem 5. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, (a) and (b) below.

(a) (i) and (ii) below are equivalent.
  (i) There exists an effective numbering $\psi : \mathbb{N} \to S$ via γ in which krt holds.
  (ii) (D, γ, S) is recursively complete.
(b) (i)-(iii) below are equivalent.
  (i) There exists an effective numbering $\psi : \mathbb{N} \to S$ via γ in which krt does *not* hold.
  (ii) $D \neq \{\perp\}$.
  (iii) $S \neq \{\perp\}$.

Proof. Let (D, γ, S) be fixed.

(a)[(i) $\Rightarrow$ (ii)]: (By contrapositive.) Suppose that (D, γ, S) is *not* recursively complete. Thus, there exists a computably enumerable $A \subseteq \mathbb{N}$ such that $\gamma(A)$ is directed and

$$\bigsqcup \gamma(A) \notin S. \tag{28}$$

Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function such that, for each s,

$$(\gamma \circ f)(s) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } A \text{ listed in fewer than } s \text{ steps.} \tag{29}$$

Clearly, $\gamma \circ f$ is monotone non-decreasing and $\bigsqcup \gamma(A) = \bigsqcup\{(\gamma \circ f)(s) \mid s \in \mathbb{N}\}$. Now, by way of contradiction, let $\psi : \mathbb{N} \to S$ be an effective numbering via γ in which krt holds. Let $\lambda s, p \centerdot \psi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ \psi^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq \psi(p)\} \text{ listed in fewer than } s \text{ steps.} \tag{30}$$

Clearly, for each p, the function $\lambda s \centerdot (\gamma \circ \psi^s)(p)$ is monotone non-decreasing and $\psi(p) = \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\}$. Let $\lambda s, p \centerdot g^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each s and p,

$$g^0(p) = f(0); \tag{31}$$

$$g^{s+1}(p) = \begin{cases} f(s+1), & \text{if } (\gamma \circ \psi^s)(p) = (\gamma \circ g^s)(p); \\ g^s(p), & \text{otherwise.} \end{cases} \tag{32}$$

Clearly, $\lambda s, p \centerdot g^s(p)$ is computable. Furthermore, since f is monotone non-decreasing, for each p, the function $\lambda s \centerdot (\gamma \circ g^s)(p)$ is monotone non-decreasing. Consider the following cases.

Case $(\forall p)$[the set $\{s \mid (\gamma \circ \psi^s)(p) = (\gamma \circ g^s)(p)\}$ is finite]. By Proposition 4,

$$g \stackrel{\text{def}}{=} \lambda p \centerdot \bigsqcup\{(\gamma \circ g^s)(p) \mid s \in \mathbb{N}\} \tag{33}$$

is effective via γ. Furthermore, by the case and (32), $\mathrm{rng}(g) \subseteq \mathrm{rng}(\gamma) \subseteq S$. Thus, by krt in ψ, there exists an e such that $\psi(e) = g(e)$. Let s_0 be such that

$$(\forall s \geq s_0)[(\gamma \circ \psi^s)(e) \neq (\gamma \circ g^s)(e)]. \tag{34}$$

Clearly, by (32),

$$(\forall s \geq s_0)[g^s(e) = g^{s_0}(e)]. \tag{35}$$

Thus,

$$g(e) = (\gamma \circ g^{s_0})(e). \tag{36}$$

Next, note that the set $B = \{(\gamma \circ \psi^s)(p) \mid s \geq s_0\}$ is directed, $\psi(e) = \bigsqcup B$, and $g(e) \sqsubseteq \bigsqcup B$. Thus, since $g(e)$ is compact in D (by (36)), there exists an $s_1 \geq s_0$ such that

$$g(e) \sqsubseteq (\gamma \circ \psi^{s_1})(e). \tag{37}$$

Furthermore,

$$(\gamma \circ \psi^{s_1})(e) \sqsubseteq \psi(e) \sqsubseteq g(e). \tag{38}$$

Thus,

$$\begin{aligned}(\gamma \circ \psi^{s_1})(e) &= g(e) && \{\text{by (37) and (38)}\}\\ &= (\gamma \circ g^{s_0})(e) && \{\text{by (36)}\}\\ &= (\gamma \circ g^{s_1})(e) && \{\text{by (35) and } s_1 \geq s_0\},\end{aligned}$$

which contradicts (34).

CASE $(\exists p)$[the set $\{s \mid (\gamma \circ \psi^s)(p) = (\gamma \circ g^s)(p)\}$ is infinite]. Then, since $\lambda s.(\gamma \circ \psi^s)(p)$ and $\lambda s.(\gamma \circ g^s)(p)$ are monotone non-decreasing,

$$\psi(p) = \bigsqcup\{(\gamma \circ g^s)(p) \mid s \in \mathbb{N}\}. \tag{39}$$

Furthermore, from the case, it follows that $\{s \mid g^s(p) = f(s)\}$ is infinite. Thus, since $\lambda s.(\gamma \circ g^s)(p)$ and $\gamma \circ f$ are monotone non-decreasing,

$$\bigsqcup\{(\gamma \circ g^s)(p) \mid s \in \mathbb{N}\} = \bigsqcup\{(\gamma \circ f)(s) \mid s \in \mathbb{N}\}. \tag{40}$$

Finally,

$$\begin{aligned}\psi(p) &= \bigsqcup\{(\gamma \circ g^s)(p) \mid s \in \mathbb{N}\} && \{\text{by (39)}\}\\ &= \bigsqcup\{(\gamma \circ f)(s) \mid s \in \mathbb{N}\} && \{\text{by (40)}\}\\ &= \bigsqcup \gamma(A) && \{\text{by the choice of } f\}\\ &\notin S && \{\text{by (28)}\}\end{aligned}$$

— a contradiction.

(a)[(ii) $\Rightarrow$ (i)]: Suppose that (D, γ, S) is recursively complete. Let $\lambda s, p.\psi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each s and p, $\psi^s(p)$ is determined as follows.

- Let $i_0, ..., i_{n-1}$ be the elements of $\{i \mid \langle i, p\rangle \in W_p\}$ listed in fewer than s steps. Let $m \leq n$ be *greatest* such that $\{\gamma(i_0), ..., \gamma(i_{m-1})\}$ has an upper-bound in D. Output the least j such that $\gamma(j) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{m-1})\}$.

Clearly, $\lambda s, p.\psi^s(p)$ is computable, and, for each p, the function $\lambda s.(\gamma \circ \psi^s)(p)$ is monotone non-decreasing. Thus, by Proposition 4,

$$\psi \overset{\text{def}}{=} \lambda p.\bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\} \tag{41}$$

is effective via γ. It is straightforward to show that $S \subseteq \text{rng}(\psi)$. Furthermore, since (D, γ, S) is recursively complete, $\text{rng}(\psi) \subseteq S$. Thus, ψ is an effective numbering of (D, γ, S). To show that krt holds in ψ, let effective $f : \mathbb{N} \to S$ via γ be fixed. Let e be such that

$$W_e = \{\langle i, p\rangle \mid \gamma(i) \sqsubseteq f(p)\}. \tag{42}$$

Clearly, such an e exists. Furthermore,

$$\psi(e) = \bigsqcup\{\gamma(i) \mid \langle i, e\rangle \in W_e\} = \bigsqcup\{\gamma(i) \mid \gamma(i) \sqsubseteq f(e)\} = f(e). \quad (43)$$

(b)[(i) $\Rightarrow$ (ii)]: (By contrapositive.) Suppose that $D = \{\bot\}$. Then, clearly, there is only one effective $f : \mathbb{N} \to S$ via γ, namely, $\lambda p.\bot$. Thus, for any effective numbering $\psi : \mathbb{N} \to S$ via γ, and any e, $\psi(e) = \bot = f(e)$.

(b)[(ii) $\Rightarrow$ (iii)]: Suppose that $D \neq \{\bot\}$. Then, it is straightforward to show that $\{\bot\} \subset \mathsf{K}(D)$. (The proof is similar to that of Proposition 9($\Leftarrow$) below.) Thus, since $\mathsf{K}(D) \subseteq S$, $S \neq \{\bot\}$.

(b)[(iii) $\Rightarrow$ (i)]: Suppose that $S \neq \{\bot\}$. It is straightforward to construct an effective numbering $\psi : \mathbb{N} \to S$ via γ so that, for each p,

$$\psi(p) = \bot \iff p = 0. \quad (44)$$

Let $x_0 \in S$ be such that $x_0 \neq \bot$. Let $f : \mathbb{N} \to S$ be such that

$$f(p) = \begin{cases} x_0, & \text{if } p = 0; \\ \bot, & \text{otherwise.} \end{cases} \quad (45)$$

Clearly, f is effective via γ, and, for each p, $\psi(p) \neq f(p)$. □ (**Theorem 5**)

Theorem 7. Suppose that (D, γ, S) is a constructive Scott subdomain, and that D is discriminable. Then, for each effective numbering $\psi : \mathbb{N} \to S$ via γ, (a) and (b) below are equivalent.

(a) krt holds in ψ.
(b) ($\forall$ effective $f : \mathbb{N} \to S$ via γ)($\exists e$)[$f(e) \sqsubseteq \psi(e)$].

Proof. Let (D, γ, S) and ψ be as stated. Without loss of generality, suppose that $\gamma(0) = \bot$.

(a) $\Rightarrow$ (b): Immediate.

(b) $\Rightarrow$ (a): Let $\lambda s, p.\psi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ \psi^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq \psi(p)\} \text{ listed in fewer than } s \text{ steps.} \quad (46)$$

Clearly, for each p, the function $\lambda s.(\gamma \circ \psi^s)(p)$ is monotone non-decreasing and $\psi(p) = \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\}$. Let effective $f : \mathbb{N} \to S$ via γ be fixed. Let $\lambda s, p.f^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ f^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq f(p)\} \text{ listed in fewer than } s \text{ steps.} \quad (47)$$

Clearly, for each p, the function $\lambda s.(\gamma \circ f^s)(p)$ is monotone non-decreasing and $f(p) = \bigsqcup\{(\gamma \circ f^s)(p) \mid s \in \mathbb{N}\}$. Let $\lambda s, p.g^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each s and p,

$$g^0(p) = 0 \text{ (thus, } (\gamma \circ g^0)(p) = \bot); \tag{48}$$

$$g^{s+1}(p) = \begin{cases} i, & \text{if } (*)[(\gamma \circ \psi^s)(p) \not\sqsubseteq (\gamma \circ g^s)(p)], \text{ where } i \text{ is} \\ & \quad \textit{first found} \text{ such that } (\gamma \circ g^s)(p) \sqsubseteq \gamma(i) \\ & \quad \text{and } \{(\gamma \circ \psi^s)(p), \gamma(i)\} \text{ has no upper-} \\ & \quad \text{bound in } D; \\ f^{s+1}(p), & \text{if } \neg(*) \ \wedge \ (\gamma \circ g^s)(p) \sqsubseteq (\gamma \circ f^{s+1})(p); \\ g^s(p), & \text{otherwise.} \end{cases} \tag{49}$$

Clearly, $\lambda s, p.g^s(p)$ is computable, and, for each p, $\lambda s.(\gamma \circ g^s)(p)$ is monotone non-decreasing. Thus, by Proposition 4,

$$g \stackrel{\text{def}}{=} \lambda p.\bigsqcup\{(\gamma \circ g^s)(p) \mid s \in \mathbb{N}\} \tag{50}$$

is effective via γ. It is easily seen that, for each p,

$$g(p) \neq f(p) \ \Rightarrow \ (\exists s)[(\gamma \circ \psi^s)(p) \not\sqsubseteq (\gamma \circ g^s)(p)]. \tag{51}$$

By (b) in the statement of the theorem, there exists a p be such that

$$g(p) \sqsubseteq \psi(p). \tag{52}$$

Consider the following cases.

Case $(\exists s)[(\gamma \circ \psi^s)(p) \not\sqsubseteq (\gamma \circ g^s)(p)]$. Then, clearly, $g^{s+1}(p) = i$ for some i such that $\{(\gamma \circ \psi^s)(p), \gamma(i)\}$ has no upper-bound in D. Thus $\{\psi(p), g(p)\}$ has no upper-bound in D, contradicting (52).

Case $(\forall s)[(\gamma \circ \psi^s)(p) \sqsubseteq (\gamma \circ g^s)(p)]$. Then, clearly,

$$\psi(p) = \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\} \sqsubseteq \bigsqcup\{(\gamma \circ g^s)(p) \mid s \in \mathbb{N}\} = g(p). \tag{53}$$

Thus, by (52), $\psi(p) = g(p)$. Furthermore, by the case and (51), $g(p) = f(p)$. Thus, $\psi(p) = g(p) = f(p)$. □ (**Theorem 7**)

Theorem 8. Suppose that (D, γ, S) is a constructive Scott subdomain, and that D is *not* discriminable. Then, there exists an effective numbering $\psi : \mathbb{N} \to S$ via γ satisfying (a) and (b) below.

(a) $(\forall$ effective $f : \mathbb{N} \to S$ via $\gamma)(\exists e)[f(e) \sqsubseteq \psi(e)]$.
(b) krt does *not* hold in ψ.

Proof. Let (D, γ, S) be as stated. Since D is *not* discriminable, there exist $x_0, y_0 \in \mathsf{K}(D)$ such that

$$y_0 \not\sqsubseteq x_0 \ \wedge \ \big(\forall z \in \mathsf{K}(D)\big)[x_0 \sqsubseteq z \ \Rightarrow \ \{y_0, z\} \text{ has an upper-bound in } D]. \tag{54}$$

Without loss of generality, suppose that $\gamma(0) = \bot$ and $\gamma(1) = x_0$. If krt does not hold in *any* effective numbering of (D, γ, S), then we are done. So, suppose that ξ is effective numbering of (D, γ, S) in which krt holds. Let $\lambda s, p \centerdot \xi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ \xi^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq \xi(p)\} \text{ listed in fewer than } s \text{ steps.} \tag{55}$$

Clearly, for each p, the function $\lambda s \centerdot (\gamma \circ \xi^s)(p)$ is monotone non-decreasing and $\xi(p) = \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\}$. Let $\lambda s, p \centerdot \psi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each s and i,

$$\psi^s(0) = 1 \text{ (thus, } (\gamma \circ \psi^s)(0) = x_0); \tag{56}$$

$$\psi^s(2i+1) = \begin{cases} i, & \text{if } x_0 \sqsubseteq (\gamma \circ \xi^s)(i), \text{ where } i \text{ is such that } \gamma(i) = y_0 \sqcup (\gamma \circ \xi^s)(i); \\ \xi^s(i), & \text{otherwise;} \end{cases} \tag{57}$$

$$\psi^0(2i+2) = 0 \text{ (thus, } (\gamma \circ \psi^0)(2i+2) = \bot); \tag{58}$$

$$\psi^{s+1}(2i+2) = \begin{cases} \xi^{s+1}(i), & \text{if } x_0 \not\sqsubseteq (\gamma \circ \xi^{s+1})(i) \ \lor \ x_0 \sqsubset (\gamma \circ \xi^{s+1})(i); \\ \psi^s(2i+2), & \text{otherwise (i.e., } (\gamma \circ \xi^{s+1})(i) = x_0). \end{cases} \tag{59}$$

Clearly, $\lambda s, p \centerdot \psi^s(p)$ is computable. Furthermore, it is straightforward to show that, for each p, the function $\lambda s \centerdot (\gamma \circ \psi^s)(p)$ is monotone non-decreasing. Thus, by Proposition 4,

$$\psi \stackrel{\text{def}}{=} \lambda p \centerdot \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\} \tag{60}$$

is effective via γ.

It is also straightforward to show that, for each $z \in S$, there exists a $p \in 2\mathbb{N}$ such that $\psi(p) = z$. (More specifically: if $z = x_0$, then $p = 0$; if $z \neq x_0$, then $p \in 2\mathbb{N} + 2$.) Thus, ψ is an effective numbering of (D, γ, S).

To see that ψ satisfies (a) in the statement of the theorem, let effective $f : \mathbb{N} \to S$ via γ be fixed. Let $g : \mathbb{N} \to S$ be such that, for each i,

$$g(i) = f(2i+1). \tag{61}$$

Clearly, g is effective via γ. Thus, by krt in ξ, there exists an i such that $\xi(i) = g(i)$. Clearly, $\xi(i) \sqsubseteq \psi(2i+1)$. Thus,

$$f(2i+1) = g(i) = \xi(i) \sqsubseteq \psi(2i+1). \tag{62}$$

It remains to show that krt does *not* hold in ψ.

Claim 8.1. $(\forall i)[\psi(2i+1) \neq x_0]$.

Proof of Claim. By way of contradiction, let i be such that $\psi(2i+1) = x_0$. Then, clearly, $\psi(2i+1) = y_0 \sqcup \xi(i)$. But then, $y_0 \sqsubseteq y_0 \sqcup \xi(i) = \psi(2i+1) = x_0$, contradicting (54). □ (**Claim 8.1**)

Let $f : \mathbb{N} \to S$ be such that, for each p,

$$f(p) = \begin{cases} y_0, \text{ if } p = 0; \\ x_0, \text{ otherwise.} \end{cases} \tag{63}$$

Clearly, f is effective via γ. By Claim 8.1, $(\forall p)[\psi(p) \neq f(p)]$. □ (**Theorem 8**)

Proposition 9. Suppose that D is a Scott domain. Then, D is flat $\Leftrightarrow$ $\mathsf{K}(D)$ is flat.

Proof of Proposition. ($\Rightarrow$) Immediate. ($\Leftarrow$) Suppose that $\mathsf{K}(D)$ is flat. To show the proposition, it suffices to show that $D \subseteq \mathsf{K}(D)$. Let $y \in D$ be fixed. Since D is algebraic, the set $A = \{x \in \mathsf{K}(D) \mid x \sqsubseteq y\}$ is directed and $y = \bigsqcup A$. Since A is directed and $A \subseteq \mathsf{K}(D)$, A must be of the form: $\{\bot\}$, $\{x\}$, or $\{\bot, x\}$, for some $x \in \mathsf{K}(D)$. Clearly, for each possibility, $\bigsqcup A \in \mathsf{K}(D)$. Thus, $y \in \mathsf{K}(D)$.
□ (**Proposition 9**)

Theorem 11. Suppose that (D, γ, S) is a constructive Scott subdomain. Then, (a)-(c) below are equivalent.

(a) krt and fprt are equivalent with respect to (D, γ, S).
(b) D is flat.
(c) S is flat.

Proof. (a) $\Rightarrow$ (b): (By contrapositive.) Let (D, γ, S) be a constructive Scott subdomain such that D is *not* flat. To show this part of the theorem, it suffices to exhibit an effective numbering $\psi : \mathbb{N} \to S$ via γ in which fprt holds, and in which krt does *not* hold. By Proposition 9, $\mathsf{K}(D)$ is *not* flat. Thus, there exist $x_0, y_0 \in \mathsf{K}(D)$ such that $\bot \sqsubset x_0 \sqsubset y_0$. Without loss of generality, suppose that $\gamma(0) = \bot$ and $\gamma(1) = x_0$. Let ξ be any effective numbering of (D, γ, S). Let $\lambda s, p\,\boldsymbol{.}\,\xi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ \xi^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq \xi(p)\} \text{ listed in fewer than } s \text{ steps.} \tag{64}$$

Clearly, for each p, the function $\lambda s\,\boldsymbol{.}\,(\gamma \circ \xi^s)(p)$ is monotone non-decreasing and $\xi(p) = \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\}$. Let $\lambda s, p\,\boldsymbol{.}\,\psi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each s and i,

$$\psi^s(0) = 1 \text{ (thus, } (\gamma \circ \psi^s)(0) = x_0); \tag{65}$$

$$\psi^0(2i + 1) = 0 \text{ (thus, } (\gamma \circ \psi^0)(2i + 1) = \bot); \tag{66}$$

$$\psi^{s+1}(2i + 1) = \begin{cases} (\psi^s \circ \varphi_i^s)(2i + 1), & \text{if } \varphi_i^s(2i + 1) \neq \bot; \\ 0, & \text{otherwise;} \end{cases} \tag{67}$$

$$\psi^0(2i + 2) = 0 \text{ (thus, } (\gamma \circ \psi^0)(2i + 2) = \bot); \tag{68}$$

$$\psi^{s+1}(2i + 2) = \begin{cases} \xi^{s+1}(i), & \text{if } x_0 \not\sqsubseteq (\gamma \circ \xi^{s+1})(i) \ \vee\ x_0 \sqsubset (\gamma \circ \xi^{s+1})(i); \\ \psi^s(2i + 2), & \text{otherwise (i.e., } (\gamma \circ \xi^{s+1})(i) = x_0). \end{cases} \tag{69}$$

Clearly, $\lambda s, p \boldsymbol{.} \psi^s(p)$ is computable. Furthermore, it is straightforward to show that, for each p, the function $\lambda s \boldsymbol{.} (\gamma \circ \psi^s)(p)$ is monotone non-decreasing. Thus, by Proposition 4,

$$\psi \stackrel{\text{def}}{=} \lambda p \boldsymbol{.} \bigsqcup \{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\} \tag{70}$$

is effective via γ.

It is also straightforward to show that, for each $z \in S$, there exists a $p \in 2\mathbb{N}$ such that $\psi(p) = z$. (More specifically: if $z = x_0$, then $p = 0$; if $z \neq x_0$, then $p \in 2\mathbb{N}+2$.) Thus, ψ is an effective numbering of (D, γ, S). That fprt holds in ψ is shown by Claim 11.1 below. That krt does *not* hold in ψ is shown by Claim 11.3 below.

Claim 11.1. fprt holds in ψ.

Proof of Claim. Let computable $t : \mathbb{N} \to \mathbb{N}$ be fixed. Let i be such that $\varphi_i = t$. Since t is computable, $\varphi_i(2i+1) \neq \bot$. Clearly, by the construction of ψ, $\psi(2i+1) = (\psi \circ \varphi_i)(2i+1)$. Thus, $\psi(2i+1) = (\psi \circ t)(2i+1)$. □ (**Claim 11.1**)

Claim 11.2. The predicate $\lambda p \in 2\mathbb{N}+1 \boldsymbol{.} [\psi(p) = x_0]$ is partial computable.

Proof of Claim. Let $p \in 2\mathbb{N}+1$ be fixed. Clearly, $\psi(p) = x_0 \Leftrightarrow$ there exists a finite sequence $i_0, ..., i_n$ satisfying (a)-(c) below.

(a) $2i_0 + 1 = p$.
(b) $(\forall j < n)[\varphi_{i_j}(2i_j + 1) = 2i_{j+1} + 1]$.
(c) $\varphi_{i_n}(2i_n + 1) = 0$.

Since the existence of such a sequence is partial computable, the claim follows. □ (**Claim 11.2**)

Claim 11.3. krt does *not* hold in ψ.

Proof of Claim. Let $f : \mathbb{N} \to S$ be such that, for each p,

$$f(0) = y_0; \tag{71}$$

$$f(2i+1) = \begin{cases} y_0, \text{ if } \psi(2i+1) = x_0; \\ x_0, \text{ otherwise;} \end{cases} \tag{72}$$

$$f(2i+2) = x_0. \tag{73}$$

By Claim 11.2, f is effective via γ. Clearly, $(\forall p)[\psi(p) \neq f(p)]$. □ (**Claim 11.3**)

(b) $\Rightarrow$ (c): Immediate.

(c) $\Rightarrow$ (a): Let (D, γ, S) be a constructive Scott subdomain such that S is flat. Clearly,

$$(\forall x, y \in S)[\bot \sqsubset x \sqsubseteq y \;\Rightarrow\; x = y]. \tag{74}$$

By Proposition 10, it suffices to show that: for every effective numbering of (D, γ, S) in which krt does *not* hold, fprt also does *not* hold. If krt holds in *every* effective numbering of (D, γ, S), then we are done. So, suppose that ψ is an effective numbering of (D, γ, S) in which krt does *not* hold. Let $\lambda s, p \centerdot \psi^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ \psi^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq \psi(p)\} \text{ listed in fewer than } s \text{ steps.} \tag{75}$$

Clearly, for each p, the function $\lambda s \centerdot (\gamma \circ \psi^s)(p)$ is monotone non-decreasing and $\psi(p) = \bigsqcup\{(\gamma \circ \psi^s)(p) \mid s \in \mathbb{N}\}$. Since krt does *not* hold in ψ, there exists an effective $f : \mathbb{N} \to S$ via γ such that

$$(\forall p)[\psi(p) \neq f(p)]. \tag{76}$$

Let $\lambda s, p \centerdot f^s(p) : \mathbb{N}^2 \to \mathbb{N}$ be a computable function such that, for each s and p,

$$(\gamma \circ f^s)(p) = \bigsqcup\{\gamma(i_0), ..., \gamma(i_{n-1})\}, \text{where } i_0, ..., i_{n-1} \text{ are the elements of } \{i \mid \gamma(i) \sqsubseteq f(p)\} \text{ listed in fewer than } s \text{ steps.} \tag{77}$$

Clearly, for each p, the function $\lambda s \centerdot (\gamma \circ f^s)(p)$ is monotone non-decreasing and $f(p) = \bigsqcup\{(\gamma \circ f^s)(p) \mid s \in \mathbb{N}\}$. Let p_0 be such that $\psi(p_0) = \bot$. Let $t : \mathbb{N} \to \mathbb{N}$ be such that, for each p, $t(p)$ is determined as follows.

> Dovetail between the following two steps until one applies. (By (76), one must eventually apply.)
>
> (i) If, for some s, $(\gamma \circ f^s)(p) \neq \bot$, output q, where q is *first found* such that $(\gamma \circ f^s)(p) \sqsubseteq \psi(q)$.
>
> (ii) If, for some s, $(\gamma \circ \psi^s)(p) \neq \bot$, output p_0.

Clearly, t is computable. It remains to show that $(\forall p)[\psi(p) \neq (\psi \circ t)(p)]$. Let p be fixed, and consider the following cases.

Case [step (i) applies first in the computation of $t(p)$]. Let s and q be as in the computation of $t(p)$. By (74), $f(p) = (\gamma \circ f^s)(p)$ and $\psi(q) = (\gamma \circ f^s)(p)$. Thus, $\psi(p) \neq f(p) = (\gamma \circ f^s)(p) = \psi(q) = (\psi \circ t)(p)$.

Case [step (ii) applies first in the computation of $t(p)$]. Then, clearly, $\psi(p) \neq \bot = \psi(p_0) = (\psi \circ t)(p)$. □ (**Theorem 11**)