Independence Results for n-Ary Recursion Theorems*

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Abstract. The *n-ary first* and *second recursion theorems* formalize two distinct, yet similar, notions of *self-reference*. Roughly, the n-ary first recursion theorem says that, for any n algorithmic tasks (of an appropriate type), there exist n *partial computable functions* that use their own *graphs* in the manner prescribed by those tasks; the n-ary second recursion theorem says that, for any n algorithmic tasks (of an appropriate type), there exist n *programs* that use their own *source code* in the manner prescribed by those tasks.

Results include the following. The constructive 1-ary form of the first recursion theorem is *independent* of either 1-ary form of the second recursion theorem. The constructive 1-ary form of the first recursion theorem does *not* imply the constructive 2-ary form; *however*, the constructive 2-ary form *does* imply the constructive n-ary form, for each $n \geq 1$. For each $n \geq 1$, the *not-necessarily-constructive* n-ary form of the second recursion theorem does *not* imply the $(n+1)$-ary form.

1 Introduction

The *n-ary first* and *second recursion theorems* [Rog67, Ch. 11][1] formalize two distinct, yet similar, notions of *self-reference*. (Henceforth, we shall refer to these simply as the *first* and *second recursion theorems*.) In a sense, the first recursion theorem asserts the existence of *partial computable functions* that refer to their own *graphs*; the second recursion theorem asserts the existence of *programs* that refer to their own *source code*. Formally, each theorem asserts the existence of solutions to systems of a certain type of equation. We discuss each theorem in detail in the sections that follow.

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[1] In [Rog67], what is called the second recursion theorem in Chapter 11 proper (i.e., Theorem IV) is a *strictly weaker* pseudo-fixpoint variant of Kleene's original formulation [Ric80, Theorems 5.1 and 5.3]. The correct formulation can be found in [Rog67, page 214, problem 11-4] and in Section 1.3 herein.

(a)

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let rec fib0 = function 0 -> 0 | x -> fib1(x-1)
    and fib1 = function 0 -> 1 | x -> fib0(x-1) + fib1(x-1)
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(b)

$$\varTheta_0(\alpha_0,\alpha_1)(x) = \begin{cases} 0, & \text{if } x = 0; \\ \alpha_1(x-1), & \text{otherwise.} \end{cases}$$

$$\varTheta_1(\alpha_0,\alpha_1)(x) = \begin{cases} 1, & \text{if } x = 0; \\ \alpha_0(x-1) + \alpha_1(x-1), & \text{otherwise.} \end{cases}$$

Fig. 1. (a) A system of recursive equations in Ocaml. The functions assigned to `fib0` and `fib1` are the minimal fixpoint of $(\varTheta_0, \varTheta_1)$, where the computable operators $\varTheta_0, \varTheta_1 : \mathcal{P}^2 \to \mathcal{P}$ are as in (b).

1.1 The First Recursion Theorem

Many programming languages allow one to define functions using *recursive equations*, or systems thereof. Each programming language has its own syntactic nuances; however, a system of n such equations typically has the following form. The left-hand-side of each equation contains one of n function variables; the right-hand-side of each equation contains an expression involving some subset of those n variables. For the programming language Ocaml [INR], an example is given in Figure 1(a).

The functions *defined* by such a system of equations constitute *some* solution of that system. Depending upon the semantics of the programming language, however, there may exist systems of equations for which there are *no* solutions, and there may exist systems of equations for which there are *multiple* solutions.

The first recursion theorem asserts that, for a very natural class of equations, there will always exist a solution to a system of equations drawn from that class; in fact, there will exist a solution that is, in some sense, *simplest* among all possible solutions. The first recursion theorem applies to those systems of equations that can be expressed using *computable operators*; the *simplest* solutions of such systems are called *minimal fixpoints*. We discuss each of these topics below.

Let $\mathbb{N}$ be the set of natural numbers, $\{0, 1, 2, ...\}$. Let lowercase Roman letters, with or without decorations (e.g., a, b_0, c'), range over elements of $\mathbb{N}$, unless stated otherwise. Let $\mathcal{P}$ be the set of all partial functions mapping $\mathbb{N}$ to $\mathbb{N}$. Let lowercase Greek letters, with or without decorations (e.g., α, β_0, γ'), range over elements of $\mathcal{P}$, unless stated otherwise. Let $(F_i)_{i\in\mathbb{N}}$ be any canonical enumeration of the finite functions [Rog67, MY78]. For each n, each $\alpha_0, ..., \alpha_{n-1}$, and each $\beta_0, ..., \beta_{n-1}$, $(\alpha_0, ..., \alpha_{n-1}) \subseteq (\beta_0, ..., \beta_{n-1}) \overset{\text{def}}{\Leftrightarrow} [\alpha_0 \subseteq \beta_0 \ \wedge \ \cdots \ \wedge \ \alpha_{n-1} \subseteq \beta_{n-1}]$.

Intuitively, a *computable operator* is a mapping $\varTheta : \mathcal{P}^n \to \mathcal{P}$ $(n \geq 1)$ for which there exists an algorithm for listing the graph of the partial function $\varTheta(\alpha_0, ..., \alpha_{n-1})$ from listings of the graphs of the partial functions $\alpha_0, ..., \alpha_{n-1}$; moreover, the content of the resulting graph does *not* depend upon the enumeration order chosen for each of $\alpha_0, ..., \alpha_{n-1}$ [Rog67, §9.8].[2] Uppercase Greek letters,

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[2] Rogers [Rog67] calls the computable operators, *recursive operators*.

with or without decorations (e.g., Θ, Ψ_0, Ω'), range over computable operators, unless stated otherwise.

Computable operators have the following *monotonicity* and *continuity* properties [Rog67, page 147]. For each n, and each $\Theta : \mathcal{P}^n \to \mathcal{P}$, (a) and (b) below.

(a) *Monotonicity*: For each $\alpha_0, ..., \alpha_{n-1}$ and $\beta_0, ..., \beta_{n-1}$, if $(\alpha_0, ..., \alpha_{n-1}) \subseteq (\beta_0, ..., \beta_{n-1})$, then $\Theta(\alpha_0, ..., \alpha_{n-1}) \subseteq \Theta(\beta_0, ..., \beta_{n-1})$.
(b) *Continuity*: For each $\alpha_0, ..., \alpha_{n-1}$, and each $(x, y) \in \Theta(\alpha_0, ..., \alpha_{n-1})$, there exist $i_0, ..., i_{n-1}$ such that $(F_{i_0}, ..., F_{i_{n-1}}) \subseteq (\alpha_0, ..., \alpha_{n-1})$ and $(x, y) \in \Theta(F_{i_0}, ..., F_{i_{n-1}})$.

For each n, each $\alpha_0, ..., \alpha_{n-1}$, and each $\Theta_0, ..., \Theta_{n-1} : \mathcal{P}^n \to \mathcal{P}$, $(\alpha_0, ..., \alpha_{n-1})$ is a *fixpoint* of $(\Theta_0, ..., \Theta_{n-1}) \overset{\text{def}}{\Leftrightarrow}$

$$\begin{aligned} \alpha_0 &= \quad \Theta_0(\alpha_0, ..., \alpha_{n-1}); \\ &\vdots \\ \alpha_{n-1} &= \Theta_{n-1}(\alpha_0, ..., \alpha_{n-1}). \end{aligned} \tag{1}$$

Intuitively, an $\alpha_0, ..., \alpha_{n-1}$ as in (1) can be thought of as a collection of functions that *refer to themselves*. What each α_i *does* with the information obtained from this self/other-reference is determined by Θ_i.

For each n, each $\alpha_0, ..., \alpha_{n-1}$, and each $\Theta_0, ..., \Theta_{n-1} : \mathcal{P}^n \to \mathcal{P}$, $(\alpha_0, ..., \alpha_{n-1})$ is the *minimal* fixpoint of $(\Theta_0, ..., \Theta_{n-1}) \overset{\text{def}}{\Leftrightarrow}$ (a) and (b) below.[3]

(a) $(\alpha_0, ..., \alpha_{n-1})$ is a fixpoint of $(\Theta_0, ..., \Theta_{n-1})$.
(b) For each $\beta_0, ..., \beta_{n-1}$, if $(\beta_0, ..., \beta_{n-1})$ is a fixpoint of $(\Theta_0, ..., \Theta_{n-1})$, then $(\alpha_0, ..., \alpha_{n-1}) \subseteq (\beta_0, ..., \beta_{n-1})$.

Condition (b) gives the sense in which a minimal fixpoint represents the *simplest* possible solution to a system of recursive equations: any other solution is *more complicated* in that there are *more pairs* in the graphs of its functions.

For each n, the n-ary form of the first recursion theorem says that, for each $\Theta_0, ..., \Theta_{n-1} : \mathcal{P}^n \to \mathcal{P}$, $(\Theta_0, ..., \Theta_{n-1})$ *has* a minimal fixpoint $(\alpha_0, ..., \alpha_{n-1})$, and, moreover, each of $\alpha_0, ..., \alpha_{n-1}$ is partial computable. Thus, if a system of equations can be written in the form of (1), for some $\Theta_0, ..., \Theta_{n-1}$, then that system has a simplest possible solution, namely, the minimal fixpoint of $(\Theta_0, ..., \Theta_{n-1})$. The example given in Figure 1(a) can be written in this way, using the computable operators Θ_0 and Θ_1 of Figure 1(b).

For obvious reasons, the first recursion theorem is also called the *minimal fixpoint theorem*.

1.2 The First Recursion Theorem in Programming Systems

From a programming languages standpoint, one should care, not only that a minimal fixpoint solution exists for any given system of equations, but also that

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[3] It is straightforward to show that such a fixpoint must be *unique*; hence, we are justified in calling it *the* minimal fixpoint.

there exist *programs witnessing* that minimal fixpoint. This idea is formalized in the following paragraphs.

Let $\mathcal{PC}$ be the set of all partial *computable* functions mapping $\mathbb{N}$ to $\mathbb{N}$. An *effective programming system* (eps) [Rog67, MY78] is an onto numbering $(\psi_q)_{q\in\mathbb{N}}$ of $\mathcal{PC}$ such that $\lambda q, x \boldsymbol{.} \psi_q(x)$ is partial computable. An eps may be thought of as an abstraction of the notion of *programming language*, in the following sense. If one were to take the programs in some programming language for $\mathcal{PC}$, and number those programs, e.g., length-lexicographically, then the function which sends q to the semantics of the qth program would be an eps.

For each $n \geq 1$, we say that the *not-necessarily-constructive n-ary form of the minimal fixpoint theorem* (n-mfp) *holds in* eps $(\psi_q)_{q\in\mathbb{N}}$ $\overset{\text{def}}{\Leftrightarrow}$ for each $\Theta_0, ..., \Theta_{n-1} : \mathcal{P}^n \to \mathcal{P}$, there exist $e_0, ..., e_{n-1}$ such that

$$(\psi_{e_0}, ..., \psi_{e_{n-1}}) \text{ is the minimal fixpoint of } (\Theta_0, ..., \Theta_{n-1}). \tag{2}$$

Thus, $e_0, ..., e_{n-1}$ *witness* the minimal fixpoint of $(\Theta_0, ..., \Theta_{n-1})$ in $(\psi_q)_{q\in\mathbb{N}}$.

Intuitively, $e_0, ..., e_{n-1}$ is a collection of *programs* that have *limited* knowledge of one another. More specifically, each e_i can refer to *only* the *extensional* (synonym: *denotational*) characteristics of $e_0, ..., e_{n-1}$, i.e., their I/O behavior [Roy87].

As it turns out, n-mfp is ubiquitous.

Proposition 1. For each $n \geq 1$, and each eps $(\psi_q)_{q\in\mathbb{N}}$, n-mfp holds in $(\psi_q)_{q\in\mathbb{N}}$.

Proof of Proposition. Let n and $(\psi_q)_{q\in\mathbb{N}}$ be as stated. Let $\Theta_0, ..., \Theta_{n-1} : \mathcal{P}^n \to \mathcal{P}$ be fixed. By the first recursion theorem, $(\Theta_0, ..., \Theta_{n-1})$ has a minimal fixpoint $(\alpha_0, ..., \alpha_{n-1})$, and each of $\alpha_0, ..., \alpha_{n-1}$ is partial computable. Thus, since $(\psi_q)_{q\in\mathbb{N}}$ is an *onto* map of $\mathcal{PC}$, there exist $e_0, ..., e_{n-1}$ such that $\psi_{e_0} = \alpha_0 \wedge \cdots \wedge \psi_{e_{n-1}} = \alpha_{n-1}$. $\square$ (**Proposition 1**)

One problem with n-mfp is that it lacks *constructivity*. That is, n-mfp merely requires that the witnessing programs, $e_0, ..., e_{n-1}$, *exist*. However, it would seem reasonable to expect that one could *construct* $e_0, ..., e_{n-1}$ from (codes for) $\Theta_0, ..., \Theta_{n-1}$.

For each n, let a numbering $(\Omega_j)_{j\in\mathbb{N}}$ of the computable operators of type $\mathcal{P}^n \to \mathcal{P}$ be *effective* $\overset{\text{def}}{\Leftrightarrow}$ the predicate $\lambda i, j, i_0, ..., i_{n-1} \boldsymbol{.} [F_i \subseteq \Omega_j(F_{i_0}, ..., F_{i_{n-1}})]$ is partial computable.[4] Let $\langle \cdot, \cdot \rangle$ be any fixed pairing function.[5] For each x, $\langle x \rangle \overset{\text{def}}{=} x$, and, for each $x_0, ..., x_{n-1}$, where $n > 2$, $\langle x_0, ..., x_{n-1} \rangle \overset{\text{def}}{=} \big\langle x_0, \langle x_1, ..., x_{n-1} \rangle \big\rangle$.

For each $n \geq 1$, we say that the *constructive n-ary form of the minimal fixpoint theorem* (n-MFP) *holds in* eps $(\psi_q)_{q\in\mathbb{N}}$ $\overset{\text{def}}{\Leftrightarrow}$ there exist computable functions $\mu_0, ..., \mu_{n-1} : \mathbb{N}^n \to \mathbb{N}$, and an effective numbering $(\Omega_j)_{j\in\mathbb{N}}$ of the computable operators of type $\mathcal{P}^n \to \mathcal{P}$ such that, for each $\boldsymbol{\jmath} = (j_0, ..., j_{n-1})$, (2) holds with $e_i = \mu_i(\boldsymbol{\jmath})$ and $\Theta_i = \Omega_{j_i}$, for each $i < n$, i.e.,

$$(\psi_{\mu_0(\boldsymbol{\jmath})}, ..., \psi_{\mu_{n-1}(\boldsymbol{\jmath})}) \text{ is the minimal fixpoint of } (\Omega_{j_0}, ..., \Omega_{j_{n-1}}). \tag{3}$$

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[4] Rogers' proof of the fundamental operator theorem [Rog67, Theorem 9-XXIII] shows that such numberings exist.

[5] A *pairing function* is computable, 1-1, onto, and of type $\mathbb{N}^2 \to \mathbb{N}$ [Rog67, page 64].

(Note that capital letters are used to distinguish the constructive forms of the first recursion theorem, e.g., n-MFP, from the not-necessarily-constructive forms, e.g., n-mfp.) Intuitively, each $\jmath = (j_0, ..., j_{n-1})$ names a system of equations, i.e.,

$$\begin{aligned} \alpha_0 &= \quad \Omega_{j_0}(\alpha_0, ..., \alpha_{n-1}); \\ &\vdots \\ \alpha_{n-1} &= \Omega_{j_{n-1}}(\alpha_0, ..., \alpha_{n-1}). \end{aligned} \tag{4}$$

The functions $\mu_0, ..., \mu_{n-1}$ find the simplest possible solution of that system, in the sense of (3).

Unlike n-mfp, there *do* exist epses in which n-MFP does *not* hold, for each $n \geq 1$. (See Theorem 10 below, for example.) This leads one to ask: what can be said of those epses in which n-MFP holds? What can be said of those epses in which n-MFP does *not* hold? We revisit these questions in Section 2.

1.3 The Second Recursion Theorem

While the first recursion theorem is about *partial computable functions* that refer to their own *graphs*, the second recursion theorem is about *programs* that refer to their own *source code*. Formally: for each $n \geq 1$, the *n-ary form of the second recursion theorem* (n-krt)[6] *holds in* eps $(\psi_q)_{q\in\mathbb{N}} \overset{\text{def}}{\Leftrightarrow}$ for each $\alpha_0, ..., \alpha_{n-1} \in \mathcal{PC}$, there exist $e_0, ..., e_{n-1}$ such that

$$\begin{aligned} \psi_{e_0} &= \quad \alpha_0(\langle e_0, ..., e_{n-1}, \cdot\rangle); \\ &\vdots \\ \psi_{e_{n-1}} &= \alpha_{n-1}(\langle e_0, ..., e_{n-1}, \cdot\rangle). \end{aligned} \tag{5}$$

The above can be interpreted as follows. Each e_i constructs *copies* of $e_0, ..., e_{n-1}$ — including e_i, itself. Then, e_i performs its associated task, α_i, using these self/other-copies.

These self/other-copies provide e_i *complete*, *low-level* knowledge of $e_0, ..., e_{n-1}$. As such, e_i is able to reflect upon the *intensional* (synonym: *connotational*) characteristics of $e_0, ..., e_{n-1}$, e.g., their sizes, runtimes, memory usage, etc. Of course, by simulating $e_0, ..., e_{n-1}$, it is possible for e_i to reflect upon their *extensional* characteristics as well [Roy87].

The proof of Theorem 5 in [Cas76] (Theorem 2 below) features a nice application of 2-krt. We give some highlights of the proof below. Let $(\varphi_p)_{p\in\mathbb{N}}$ be any standard numbering of $\mathcal{PC}$.[7] For each $p \in \mathbb{N}$, let W_p be the domain of φ_p. Thus, $(W_p)_{p\in\mathbb{N}}$ is a (standard) numbering of the computably enumerable (ce) sets [Rog67].

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[6] The 'k' in n-krt is for "Kleene". The 1-ary forms of the two recursion theorems are due to him. The generalized n-ary first recursion theorem is due to Manna, *et al.* [MNV72, pages 30 and 31]. The 2-ary form of the second recursion theorem follows essentially from Smullyan's [Smu61, page 75, Theorem 5]. The generalized n-ary second recursion theorem appears to be a folk theorem.

[7] Any standard numbering is *acceptable*. As such, n-krt holds in such a numbering, for each $n \geq 1$. (See the discussion surrounding Theorem 5 below.)

Theorem 2 (Case [Cas76, Theorem 5]). There is *no* algorithm to extend a computable partial order to ce total order, in the following sense. There is *no* computable function $f : \mathbb{N} \to \mathbb{N}$ such that, for each x and y, if

- φ_x is a characteristic function for a finite set A,[8]
- φ_y is a characteristic function for a set $R \subseteq A \times A$, *and*
- the transitive closure of R is a partial order on A,

then $W_{f(x,y)}$ is a *total* order on A which includes the transitive closure of R.[9]

The proof of Theorem 2 begins by supposing that such an f exists. Two programs, e_0 and e_1, are then obtained via an application of 2-krt. Intuitively, e_0 plays the role of x in Theorem 2, while e_1 plays the role of y. Each program: (1) constructs copies of both *itself* and the *other*, (2) computes $f(e_0, e_1)$ using these self/other-copies, and then (3) begins listing $W_{f(e_0,e_1)}$. By reacting to the pairs so listed, e_0 and e_1 are able to cause f to *fail* to meet its specification, thereby obtaining a contradiction.

Another interesting application of 2-krt appears in the proof of [WZ95, Theorem 3].

Like n-mfp (Section 1.2), there is no constructivity in the definition of n-krt. Unlike n-mfp, however, there *do* exist epses in which n-krt does *not* hold, for each $n \geq 1$.[10]

Nearly every mainstream programming language supports recursive equations of the form of (1). In this sense, the first recursion theorem is explicitly *built-in* to such programming languages. No mainstream programming language seems to have the *second* recursion theorem so *explicitly* built-in, however. We recommend that such programming languages *be* developed since they would have applications, e.g., for self-modeling in artificial intelligence, as suggested by [Ada06, BZL06, Con07, Sch07].[11]

1.4 *Constructive* Forms of the Second Recursion Theorem

The second recursion theorem has constructive forms similar to those presented for the first recursion theorem in Section 1.2 (i.e., n-MFP). For each $n \geq 1$, the *n-ary form of the relatively constructive second recursion theorem* (n-RelKRT) *holds in* eps $(\psi_q)_{q\in\mathbb{N}} \overset{\text{def}}{\Leftrightarrow}$ there exist computable functions $r_0, ..., r_{n-1} : \mathbb{N}^n \to \mathbb{N}$, and an eps $(\xi_p)_{p\in\mathbb{N}}$ such that, for each $\boldsymbol{p} = (p_0, ..., p_{n-1})$, (5) holds with $e_i = r_i(\boldsymbol{p})$

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[8] A *characteristic function for a set A* is a (total) function $g : \mathbb{N} \to \{0, 1\}$ such that $(\forall x)[g(x) = 1 \;\Leftrightarrow\; x \in A]$.

[9] The action of f in Theorem 2 can be seen as a form of *topological sort* [Knu73].

[10] This follows from Riccardi's [Ric80, Theorem 3.9] (also [Ric81, Theorem 2.9]) and the existence of Friedberg numberings [Fri58, Kum90], for example.

[11] We would also like to understand *mathematically* the usefulness and possible profundity of perfect n-ary self/other-modeling and self/other-knowledge; hence, a future project is to *insightfully* characterize, for each $n \geq 1$, the epses in which n-krt holds.

and $\alpha_i = \xi_{p_i}$, for each $i < n$, i.e.,

$$\begin{aligned} \psi_{r_0(\boldsymbol{p})} &= \quad \xi_{p_0}\big(\langle r_0(\boldsymbol{p}), ..., r_{n-1}(\boldsymbol{p}), \cdot\rangle\big); \\ &\vdots \\ \psi_{r_{n-1}(\boldsymbol{p})} &= \xi_{p_{n-1}}\big(\langle r_0(\boldsymbol{p}), ..., r_{n-1}(\boldsymbol{p}), \cdot\rangle\big). \end{aligned} \tag{6}$$

In (6), $(\psi_q)_{q\in\mathbb{N}}$ is an eps for representing *self-referential programs* (e.g., $r_0(\boldsymbol{p})$, ..., $r_{n-1}(\boldsymbol{p})$), while $(\xi_p)_{p\in\mathbb{N}}$ is an eps for representing programs for *tasks* (e.g., $p_0, ..., p_{n-1}$).

The 1-ary form of RelKRT was introduced in [CM07a]. Therein, it was shown that 1-krt and 1-RelKRT are *equivalent*, in the following sense.

Theorem 3 (Case, Moelius [CM07a, Theorem 2]). For each eps $(\psi_q)_{q\in\mathbb{N}}$, 1-krt holds in $(\psi_q)_{q\in\mathbb{N}} \Leftrightarrow$ 1-RelKRT holds in $(\psi_q)_{q\in\mathbb{N}}$.

Thus, for any eps containing self-referential programs, there exists *some* effective numbering of all algorithmic tasks from which those self-referential programs can be found constructively.

A special case of RelKRT that has been considered frequently in the literature (e.g., in [Ric80, Ric81, Roy87]) is the following. For each $n \geq 1$, the *constructive n-ary form of the second recursion theorem* (n-KRT) *holds in* eps $(\psi_q)_{q\in\mathbb{N}} \overset{\text{def}}{\Leftrightarrow}$ there exist computable functions $r_0, ..., r_{n-1} : \mathbb{N}^n \to \mathbb{N}$ such that, for each $\boldsymbol{q} = (q_0, ..., q_{n-1})$, (6) holds with $(\psi_q)_{q\in\mathbb{N}} = (\xi_p)_{p\in\mathbb{N}}$, i.e.,

$$\begin{aligned} \psi_{r_0(\boldsymbol{q})} &= \quad \psi_{q_0}\big(\langle r_0(\boldsymbol{q}), ..., r_{n-1}(\boldsymbol{q}), \cdot\rangle\big); \\ &\vdots \\ \psi_{r_{n-1}(\boldsymbol{q})} &= \psi_{q_{n-1}}\big(\langle r_0(\boldsymbol{q}), ..., r_{n-1}(\boldsymbol{q}), \cdot\rangle\big). \end{aligned} \tag{7}$$

KRT is a special case of RelKRT in that the eps for representing self-referential programs (i.e., $(\psi_q)_{q\in\mathbb{N}}$), and the eps for representing programs for tasks (i.e., $(\xi_p)_{p\in\mathbb{N}}$), are the same.

In his thesis, Riccardi showed the following.

Theorem 4 (Riccardi [Ric80, Theorem 3.15], [Ric81, Theorem 2.13]). There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that $(\forall n \geq 1)[n\text{-krt holds in } (\psi_q)_{q\in\mathbb{N}}]$, but 1-KRT does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

In addition to the above, Riccardi's thesis featured another remarkable result. An eps $(\psi_q)_{q\in\mathbb{N}}$ is *acceptable* $\overset{\text{def}}{\Leftrightarrow}$ every other eps can be compiled into $(\psi_q)_{q\in\mathbb{N}}$, i.e., $(\forall \text{ eps } (\xi_p)_{p\in\mathbb{N}})(\exists \text{ computable } t : \mathbb{N} \to \mathbb{N})(\forall p)[\psi_{t(p)} = \xi_p]$ [Rog67, MY78, Ric80, Ric81, Roy87].

Theorem 5 (Riccardi [Ric80, Theorem 3.6], [Ric81, Theorem 2.6]). For each eps $(\psi_q)_{q\in\mathbb{N}}$, $(\psi_q)_{q\in\mathbb{N}}$ is acceptable $\Leftrightarrow$ 2-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$.

It can be shown that, if $(\psi_q)_{q\in\mathbb{N}}$ is an acceptable eps, then $(\psi_q)_{q\in\mathbb{N}}$ has the following desirable properties.

(a) $(\psi_q)_{q\in\mathbb{N}}$ has an implementation of every *control structure*.[12]
(b) For each $n \geq 1$, n-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$.
(c) For each $n \geq 1$, n-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$.
(d) For each $n \geq 1$, n-RelKRT holds in $(\psi_q)_{q\in\mathbb{N}}$.[13]

These observations and Riccardi's Theorem 5 above imply the following.

Corollary 6 (of Theorem 5). For each eps $(\psi_q)_{q\in\mathbb{N}}$, 2-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$ $\Leftrightarrow$ $(\forall n \geq 1)[n$-KRT holds in $(\psi_q)_{q\in\mathbb{N}}]$.

Thus, having the ability to find just *two* self-referential programs constructively for any *two* programs for tasks implies having the ability to find n self-referential programs constructively for any n programs for tasks, *provided* that the two varieties of program reside in the same eps.

What if one were allowed to program the tasks in some *other* eps? That is, if one were to replace 2-KRT by 2-RelKRT in Theorem 5, would the result still hold? We answer this question in the *affirmative* in Section 2.

1.5 Organization

In Sections 2 and 3, we explore the relationships among the forms of the recursion theorems mentioned above. In Section 2, we focus, primarily, on the constructive forms of the two recursion theorems; in Section 3 we focus on the second recursion theorem. Complete proofs of all theorems can be found in the appendix.

For the remainder, we focus exclusively on effective numberings of $\mathcal{PC}$ (i.e., epses). However, it is worth mentioning that the two recursion theorems also have applications to effective numberings of subsets of the *computable* functions. See, for example, [Koz80, RC94, BK08].

2 Constructive Forms of the Recursion Theorems

In this section, we explore the relationships among the constructive forms of the two recursion theorems (i.e., MFP, RelKRT, and KRT). Our main results of this section (summarized in Figure 2) are:

- 2-MFP entails acceptability (Theorem 7).
- 2-RelKRT entails acceptability (Theorem 9).
- 1-KRT does *not* entail 1-MFP (Theorem 10).
- 1-MFP entails *neither* 1-krt (Theorem 11) *nor* 2-MFP (Corollary 12).

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[12] See [Ric80, Ric81] for an explanation of this result.

[13] In fact, (d) follows from (c).

Since acceptability yields all of the properties considered herein, we must restrict attention to *non*-acceptable epses in order to understand the interrelatedness of these properties. epses corresponding to standard, general purpose programming languages (e.g., Lisp, C++, or Ocaml) are acceptable. However, independence proofs (e.g., in set theory and herein) often require the construction of pathological models.

$$\begin{tikzcd}
\textsf{2-MFP} \arrow[rr] \arrow[d, leftrightarrow] & & \textsf{1-MFP} \\
\textsf{2-KRT} \arrow[r] \arrow[d, leftrightarrow] & \textsf{1-KRT} \arrow[r] & \textsf{1-krt} \arrow[d, leftrightarrow] \\
\textsf{2-RelKRT} \arrow[rr] & & \textsf{1-RelKRT}
\end{tikzcd}$$

Fig. 2. A summary of the main results of Section 2. Arrows indicate entailment relationships. The reflexive-transitive closure of the above diagram represents *all* of the entailment relationships that hold among the forms of the recursion theorem appearing therein.

Theorem 7 and its corollary are our first main results.

Theorem 7. For each eps $(\psi_q)_{q\in\mathbb{N}}$, if 2-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$, then $(\psi_q)_{q\in\mathbb{N}}$ is acceptable.

Corollary 8 (of Theorem 7). For each eps $(\psi_q)_{q\in\mathbb{N}}$, 2-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$ $\Leftrightarrow$ $(\forall n \geq 1)[n\text{-MFP}$ holds in $(\psi_q)_{q\in\mathbb{N}}]$.

Thus, having the ability to find a minimal fixpoint solution constructively for *any two* recursive equations implies having the ability to find a minimal fixpoint solution constructively for *any number of* recursive equations.

Theorem 9 just below says that 2-RelKRT entails acceptability. Recall that Riccardi's Theorem 5 above said: having the ability to find just *two* self-referential programs constructively for any *two* programs for tasks implies having the ability to find n self-referential programs constructively for any n programs for tasks, *provided* that the two varieties of program reside in the same eps. Theorem 9 says that Riccardi's result still holds even if the two varieties of program are allowed to reside in *distinct* epses. The proof of Theorem 9 is similar to Riccardi's proof of Theorem 5 (see [Ric80, Theorem 3.6] or [Ric81, Theorem 2.6]).

Theorem 9. For each eps $(\psi_q)_{q\in\mathbb{N}}$, if 2-RelKRT holds in $(\psi_q)_{q\in\mathbb{N}}$, then $(\psi_q)_{q\in\mathbb{N}}$ is acceptable.

Theorem 10 just below says that 1-KRT does *not* entail 1-MFP. This result was a surprise to us. MFP provides its witnessing programs access to *only* their extensional characteristics. krt, on the other hand, provides its witnessing programs access to *both* their intensional *and* extensional characteristics. (See the discussions following (2) in Section 1.2, and (5) in Section 1.3). Thus, we had expected an entailment relationship to hold between n-krt and n-MFP, and, thus, between n-KRT and n-MFP. As Theorem 10 asserts, however, this is not the case. Understanding *why* is the subject of future research.[14] The proof of Theorem 10 is a *finite-injury priority argument* [Rog67, page 166].

Theorem 10. There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$, but 1-MFP does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

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[14] Perhaps this has something to do with minimal versus *non*-minimal fixpoints.

$$\mathsf{1\text{-}KRT} \longrightarrow \mathsf{1\text{-}krt} \longleftarrow \mathsf{2\text{-}krt} \longleftarrow \mathsf{3\text{-}krt} \longleftarrow \cdots$$

Fig. 3. A summary of the main results of Section 3. Arrows indicate entailment relationships. The reflexive-transitive closure of the above diagram represents *all* of the entailment relationships that hold among the forms of the recursion theorem appearing therein.

Theorem 11 just below says that 1-MFP does *not* entail 1-krt.

Theorem 11. There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$, but 1-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

Corollary 12 just below says that 1-MFP does *not* entail 2-MFP. Recall that Corollary 8 above said: 2-MFP entails n-MFP, for each $n \geq 1$. Corollary 12 says, essentially, that this collapse which occurs *upward* of $n = 2$ does *not* extend *below* $n = 2$.

Corollary 12 (of Theorems 7 and 11). There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$, but 2-MFP does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

We have not investigated whether n-krt entails 1-MFP, for $n \geq 2$. However, we conjecture: there exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that $(\forall n \geq 1)[n$-krt holds in $(\psi_q)_{q\in\mathbb{N}}]$, but 1-MFP does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$. We also think it would be interesting to explore properties complementary to 1-MFP, in the spirit of [CM07b].

3 The Second Recursion Theorem

In this section, we explore the relationships among various forms of the second recursion theorem (i.e., KRT and krt). Our main results of this section (summarized in Figure 3) are:

- For each $n \geq 1$, n-krt does *not* entail $(n + 1)$-krt (Theorem 13).
- 1-KRT does *not* entail 2-krt (Theorem 14).

Theorem 13 just below says that, for each $n \geq 1$, n-krt does *not* entail $(n+1)$-krt. Thus, the existence of self-referential programs for any n algorithmic tasks does *not* imply the existence of self-referential programs for any $n + 1$ algorithmic tasks. The proof of Theorem 13 is a finite-injury priority argument.

Theorem 13. For each $n \geq 1$, there exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that n-krt holds in $(\psi_q)_{q\in\mathbb{N}}$, but $(n + 1)$-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

Theorem 14 just below says that 1-KRT does *not* entail 2-krt. Thus, having the ability to find *one* self-referential program *constructively* for any *one* algorithmic task does *not* imply having the ability to find *two* self-referential programs — *constructively* or *otherwise* — for any *two* algorithmic tasks. The proof of Theorem 14 is a finite-injury priority argument.

Theorem 14. There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$, but 2-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

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Appendix

A Preliminaries

Computability-theoretic concepts not covered below are treated in [Rog67].

Recall that $\mathbb{N}$ denotes the set of natural numbers,$\{0, 1, 2, ...\}$, and that lowercase Roman letters, with or without decorations (e.g., a, b_0, c'), range over elements of $\mathbb{N}$, unless stated otherwise. Uppercase Roman letters, with or without decorations (e.g., A, B_0, C'), range over subsets of $\mathbb{N}$, unless stated otherwise. In several places, we use '$-$' to denote an anonymous variable ranging over $\mathbb{N}$.

Recall that $\mathcal{P}$ denotes the set of all partial functions mapping $\mathbb{N}$ to $\mathbb{N}$, and that lowercase Greek letters, with or without decorations (e.g., α, β_0, γ'), range over elements of $\mathcal{P}$, unless stated otherwise. Further recall that uppercase Greek letters, with or without decorations (e.g., $\varTheta$, $\varPsi_0$, $\varOmega'$), range over computable operators, unless stated otherwise.

For each α and x, $\alpha(x){\downarrow}$ denotes that $\alpha(x)$ converges; $\alpha(x){\uparrow}$ denotes that $\alpha(x)$ diverges.[15] For each α, $\mathrm{dom}(\alpha) \overset{\mathrm{def}}{=} \{x \mid \alpha(x){\downarrow}\}$; $\mathrm{rng}(\alpha) \overset{\mathrm{def}}{=} \{y \mid (\exists x)[\alpha(x) = y]\}$. We identify each partial function, α, with its graph, $\{(x, y) \mid \alpha(x) = y\}$. So, for example, $\emptyset$ denotes the everywhere divergent function. We use $\uparrow$ to denote the value of a divergent computation.

For each n, and each $x_0, ..., x_{n-1}$, $\pi^n_{i+1}(\langle x_0, ..., x_{n-1}\rangle) \overset{\mathrm{def}}{=} x_i$.

The following function will be of use later on. For each ℓ, i, and x, let $\mathrm{out} : \mathbb{N}^2 \to \mathbb{N}$ be as follows.

$$\mathrm{out}(\langle \ell, 0\rangle, x) = x; \tag{8}$$

$$\mathrm{out}(\langle \ell, i+1\rangle, x) = \mathrm{out}\big(\langle \ell, i\rangle, \langle\langle \ell, i+1\rangle, x\rangle\big). \tag{9}$$

So, for example, for each ℓ and x,

$$\mathrm{out}(\langle \ell, 3\rangle, x) = \Big\langle \langle \ell, 1\rangle, \Big\langle \langle \ell, 2\rangle, \langle\langle \ell, 3\rangle, x\rangle\Big\rangle\Big\rangle \tag{10}$$

Lemma 1. Suppose that $(\psi_q)_{q\in\mathbb{N}}$ is such that, for each ℓ and i,

$$\psi_{\langle \ell, i\rangle} = \psi_{\langle \ell, 0\rangle} \circ \mathrm{out}(\langle \ell, i\rangle, \cdot). \tag{11}$$

Then, (a)-(c) below.

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[15] For each α and x, $\alpha(x)$ *converges* iff there exists *some* y such that $\alpha(x) = y$; $\alpha(x)$ *diverges* iff there is *no* y such that $\alpha(x) = y$. If α is *partial computable*, and x is such that $\alpha(x)$ diverges, then one can imagine that a program associated with α *goes into an infinite loop* on input x.

(a) If $(\psi_q)_{q\in\mathbb{N}}$ is an eps, then 1-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$.
(b) For each ℓ, i, and i', if $i \le i'$ and $\mathrm{dom}(\psi_{\langle \ell,i'\rangle})$ is infinite, then $\mathrm{dom}(\psi_{\langle \ell,i\rangle})$ is infinite.
(c) For each ℓ, i, and i', if $i \le i'$, then $\mathrm{rng}(\psi_{\langle \ell,i'\rangle}) \subseteq \mathrm{rng}(\psi_{\langle \ell,i\rangle})$.

Proof. The proof of each part is straightforward. For part (a), the witnessing function can be taken to be $\lambda\langle \ell,i\rangle \,\boldsymbol{.}\, \langle \ell,i+1\rangle$. □ (**Lemma 1**)

B Proofs

Theorem 7. For each eps $(\psi_q)_{q\in\mathbb{N}}$, if 2-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$, then $(\psi_q)_{q\in\mathbb{N}}$ is acceptable.

Proof. Let $(\psi_q)_{q\in\mathbb{N}}$ be as stated. Let $\mu_0, \mu_1 : \mathbb{N}^2 \to \mathbb{N}$ and $(\varOmega_j)_{j\in\mathbb{N}}$ witness 2-MFP in $(\psi_q)_{q\in\mathbb{N}}$. Let $(\varphi_p)_{p\in\mathbb{N}}$ be any acceptable eps. Let j_0 be such that, for each α_0 and α_1,

$$\varOmega_{j_0}(\alpha_0,\alpha_1) = \begin{cases} \varphi_{\alpha_1(0)}, & \text{if } \alpha_1(0) \text{ converges;} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{12}$$

Let $t : \mathbb{N} \to \mathbb{N}$ be such that, for each p,

$$t(p) = \mu_0(j_0, j_1), \text{ where } j_1 \text{ is \textit{first found} such that } \varOmega_{j_1}(\emptyset,\emptyset)(0) = p. \tag{13}$$

Clearly, t is computable. To complete the proof, it then suffices to show that, for each p, $\psi_{t(p)} = \varphi_p$. Let p be fixed. Let j_1 be that which is selected in the computation of $t(p)$. Thus, $\varOmega_{j_1}(\emptyset,\emptyset)(0) = p$. Let (α_0,α_1) be the minimal fixpoint of $(\varOmega_{j_0}, \varOmega_{j_1})$. Note that

$$\begin{aligned} (0,p) &\in \varOmega_{j_1}(\emptyset,\emptyset) && \{\text{by the choice of } j_1\} \\ &\subseteq \varOmega_{j_1}(\alpha_0,\alpha_1) && \{\text{by the monotonicity of } \varOmega_{j_1}\} \\ &= \alpha_1 && \{\text{because } (\alpha_0,\alpha_1) \text{ is a fixpoint of } (\varOmega_{j_0},\varOmega_{j_1})\}. \end{aligned} \tag{14}$$

Thus,

$$\begin{aligned} \psi_{t(p)} &= \psi_{\mu_0(j_0,j_1)} && \{\text{by (13) and the choice of } j_1\} \\ &= \alpha_0 && \{\text{by the choices of } \mu \text{ and } \alpha_0\} \\ &= \varOmega_{j_0}(\alpha_0,\alpha_1) && \{\text{because } (\alpha_0,\alpha_1) \text{ is a fixpoint of } (\varOmega_{j_0},\varOmega_{j_1})\} \\ &= \varphi_p && \{\text{by (12) and (14)}\}. \end{aligned}$$

□ (**Theorem 7**)

Theorem 9. For each eps $(\psi_q)_{q\in\mathbb{N}}$, if 2-RelKRT holds in $(\psi_q)_{q\in\mathbb{N}}$, then $(\psi_q)_{q\in\mathbb{N}}$ is acceptable.

Proof. Let $(\psi_q)_{q\in\mathbb{N}}$ be as stated. Let $r_0, r_1 : \mathbb{N}^2 \to \mathbb{N}$ and $(\xi_j)_{j\in\mathbb{N}}$ witness 2-RelKRT in $(\psi_q)_{q\in\mathbb{N}}$. Let $f : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each j_0 and p,

$$f(j_0,p) = \begin{cases} j_1, \text{ if } (\forall \langle j_0',p'\rangle < \langle j_0,p\rangle)[f(j_0',p')\downarrow] \text{ and there} \\ \qquad \text{exists a \textit{least} } j_1 \text{ such that } r_1(j_0,j_1) \notin \\ \qquad \big\{r_1\big(j_0', f(j_0',p')\big) \mid \langle j_0',p'\rangle < \langle j_0,p\rangle\big\}; \\ \uparrow, \ \text{otherwise.} \end{cases} \tag{15}$$

Clearly, f is partial computable. That f is total follows from Claim 9.1 just below.

Claim 9.1. For each j_0, $\{r_1(j_0, j_1) \mid j_1 \in \mathbb{N}\}$ is infinite.

Proof of Claim. Let j_0 be fixed. Let j_1 and j_1' be any two ξ-programs for *distinct* constant functions. Then,

$$\psi_{r_1(j_0,j_1)} = \xi_{j_1} \neq \xi_{j_1'} = \psi_{r_1(j_0,j_1')}, \tag{16}$$

and, thus, $r_1(j_0, j_1) \neq r_1(j_0, j_1')$. Since there are infinitely many constant functions, the claim follows. $\square$ (**Claim 9.1**)

Let j_0 be such that, for each e_0 and e_1,

$$\xi_{j_0}(\langle e_0, e_1, \cdot\rangle) = \begin{cases} \varphi_p, & \text{where } \langle j_0, p\rangle \text{ is } \textit{least}, \textit{ if any}, \text{ such that} \\ & \quad r_1\big(j_0, f(j_0, p)\big) = e_1; \\ \lambda x \centerdot \uparrow, & \text{otherwise.} \end{cases} \tag{17}$$

Claim 9.2. For each e_1, there exists *at most one* pair $\langle j_0, p\rangle$ such that

$$r_1\big(j_0, f(j_0, p)\big) = e_1. \tag{18}$$

Proof of Claim. By way of contradiction, let e_1 be such that, for *distinct* pairs $\langle j_0, p\rangle$ and $\langle j_0', p'\rangle$,

$$r_1\big(j_0, f(j_0, p)\big) = r_1\big(j_0', f(j_0', p')\big) = e_1. \tag{19}$$

Without loss of generality, suppose that $\langle j_0', p'\rangle < \langle j_0, p\rangle$. Then, by the definition of f,

$$r_1\big(j_0, f(j_0, p)\big) \neq r_1\big(j_0', f(j_0', p')\big) \tag{20}$$

— a contradiction. $\square$ (**Claim 9.2**)

To complete the proof of the theorem, it suffices to show that, for each p,

$$\psi_{r_0(j_0, f(j_0,p))} = \varphi_p. \tag{21}$$

Let p be fixed. For each $i \leq 1$, let $e_i = r_i\big(j_0, f(j_0, p)\big)$. By Claim 9.2,

$$\langle j_0, p\rangle \text{ is the } \textit{unique} \text{ pair such that } r_1\big(j_0, f(j_0, p)\big) = e_1. \tag{22}$$

Thus,

$$\begin{aligned} \psi_{r_0(j_0, f(j_0,p))} &= \xi_{j_0}(\langle e_0, e_1, \cdot\rangle) && \{\text{by the choices of } r_0,\ e_0, \text{ and } e_1\} \\ &= \varphi_p && \{\text{by (17) and (22)}\}. \end{aligned}$$

$\square$ (**Theorem 9**)

The proof of Theorem 10 just below is a finite-injury priority argument.

Theorem 10. There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$, but 1-$\mathsf{MFP}$ does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

Proof. $(\psi_q)_{q\in\mathbb{N}}$ is constructed in stages. It will be clear from the construction that, for each stage s, the graph of $(\psi^s_q)_{q\in\mathbb{N}}$ is decidable. Throughout the construction, the following invariant is maintained. (Recall that $\mathrm{out} : \mathbb{N}^2 \to \mathbb{N}$ was defined in (8) and (9).) For each s, ℓ, and i,

$$\psi^s_{\langle \ell,i\rangle} = \psi^s_{\langle \ell,0\rangle} \circ \mathrm{out}(\langle \ell, i\rangle, \cdot). \tag{23}$$

For bookkeeping purposes, the construction maintains a ce set S. $S^0 = \emptyset$.

Let $(\Theta_k)_{k\in\mathbb{N}}$ be any acceptable numbering of the computable operators of type $\mathcal{P} \to \mathcal{P}$.[16] For each p and ℓ, let $\mathrm{start} : \mathbb{N} \to \mathbb{N}$ and $\mathrm{rank} : \mathbb{N} \to \mathbb{N}$ be as follows.

$$\mathrm{start}(0) = 0; \tag{24}$$
$$\mathrm{start}(p+1) = \mathrm{start}(p) + p + 1.^{17} \tag{25}$$
$$\mathrm{rank}(\ell) = \max\{p \mid \mathrm{start}(p) \le \ell\}. \tag{26}$$

The construction of $(\psi_q)_{q\in\mathbb{N}}$ is given in Figure 4.

Claim 10.1. $(\psi_q)_{q\in\mathbb{N}}$ is an eps.

Proof of Claim. To show the claim, it suffices to show that, for each p, there exists an ℓ such that $\psi_{\langle \ell,0\rangle} = \varphi_p$. By way of contradiction, let p_0 be such that, for each ℓ, $\psi_{\langle \ell,0\rangle} \neq \varphi_{p_0}$. Let L be such that

$$L = \{\ell \mid \mathrm{rank}(\ell) = p_0\}. \tag{27}$$

Clearly, if, for some $\ell \in L$, the "if" clause is satisfied in infinitely many stages of the form $2\langle \ell, -\rangle$, then $\psi_{\langle \ell,0\rangle} = \varphi_{p_0}$. Thus, it must be the case that, for each $\ell \in L$, the "if" clause is is satisfied in *only finitely many* stages of the form $2\langle \ell, -\rangle$.

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[16] For each n, we say that an effective numbering $(\Theta_k)_{k\in\mathbb{N}}$ of the computable operators of type $\mathcal{P}^n \to \mathcal{P}$ is *acceptable* $\overset{\text{def}}{\Leftrightarrow}$ for every effective numbering $(\Omega_j)_{j\in\mathbb{N}}$ of the same type, $(\exists \text{ computable } t : \mathbb{N} \to \mathbb{N})(\forall j)[\Theta_{t(j)} = \Omega_j]$. The following facts are straightforward to show.

1. For each n, there exists an acceptable numbering of the computable operators of type $\mathcal{P}^n \to \mathcal{P}$.
2. If $(\Theta_k)_{k\in\mathbb{N}}$ is an acceptable numbering of the computable operators of type $\mathcal{P}^n \to \mathcal{P}$ ($n \ge 1$), then, for each effective numbering $(\Omega_j)_{j\in\mathbb{N}}$ of the same type, $(\exists \text{ computable, 1-1 } t : \mathbb{N} \to \mathbb{N})(\forall j)[\Theta_{t(j)} = \Omega_j]$.
3. If $\mu : \mathbb{N} \to \mathbb{N}$ and $(\Omega_j)_{j\in\mathbb{N}}$ witness 1-MFP in $(\psi_q)_{q\in\mathbb{N}}$ *and* $(\Omega_j)_{j\in\mathbb{N}}$ is acceptable, then $(\psi_q)_{q\in\mathbb{N}}$ is an acceptable eps.

[17] The proof of Theorem 10 can be viewed as a *finite-injury priority argument* [Rog67, page 166] with one set of requirements being: $R_p \Leftrightarrow (\exists \ell)[\psi_{\langle \ell,0\rangle} = \varphi_p]$. A requirement R_p is *injured* whenever a pair is put into the graph of $\psi_{\langle \ell,0\rangle}$, for some $\ell \in \{\ell \mid \mathrm{start}(p) \le \ell < \mathrm{start}(p+1)\}$, in a stage of the form $2\langle p, -\rangle + 1$. In this sense, $\mathrm{start}(p+1) - \mathrm{start}(p)$ is a (computable) upper-bound in the number of injuries sustained by R_p.

Perform stages $s = 0, 1, ...,$ successively, as follows.

STAGE $s = 2\langle \ell, -\rangle$. If $\psi^s_{\langle \ell,0\rangle} \subseteq \varphi^s_{\mathrm{rank}(\ell)}$, then set $\psi^{s+1}_{\langle \ell,0\rangle} = \varphi^{s+1}_{\mathrm{rank}(\ell)}$, and, for each $i \geq 1$, set $\psi^{s+1}_{\langle \ell,i\rangle} = \psi^{s+1}_{\langle \ell,0\rangle} \circ \mathrm{out}(\langle \ell, i\rangle, \cdot)$.

STAGE $s = 2\langle p, -\rangle + 1$. Determine whether there exists a $k < s$ such that each of conditions (a)-(f) below is satisfied.

(a) $(0,0) \in \Theta^s_k\big(\{(0,0)\}\big)$.
(b) $(0,1) \in \Theta^s_k\big(\{(0,1)\}\big)$.
(c) $\varphi^s_p(k)\!\downarrow$.
(d) $\psi^s_{\langle \ell,i\rangle}(0)\!\uparrow$, where $\langle \ell, i\rangle = \varphi^s_p(k)$.
(e) $\mathrm{rank}(\ell) > p$, where $\langle \ell, i\rangle = \varphi^s_p(k)$.
(f) $p \notin S^s$.

If such a k exists, then let $\langle \ell, i\rangle$ be as in conditions (d) and (e) just above, and perform steps (i)-(iii) below. (Claim 10.5 shows that steps (i) and (ii) cause $(0,0) \in \psi^{s+1}_{\langle \ell,i\rangle}$.)

(i) Set $\big(\mathrm{out}(\langle \ell, i\rangle, 0), 0\big) \in \psi^{s+1}_{\langle \ell,0\rangle}$.
(ii) For each $i' \geq 1$, update $\psi^{s+1}_{\langle \ell,i'\rangle}$ as in a stage of the form $2\langle \ell, -\rangle$.
(iii) Set $p \in S^{s+1}$.

Fig. 4. The construction of $(\psi_q)_{q\in\mathbb{N}}$ in the proof of Theorem 10.

Note that, for each $\ell \in L$, there are two ways that a pair may be put into the graph of $\psi_{\langle \ell,0\rangle}$: the first is during a stage of the form $2\langle \ell, -\rangle$; the second is during a stage of the form $2\langle p, -\rangle + 1$, for some p satisfying

$$p < \mathrm{rank}(\ell) \ (= p_0). \tag{28}$$

Clearly, the first way cannot affect the aforementioned "if" clause. Thus, it must be the case that some pair is put into the graph of $\psi_{\langle \ell,0\rangle}$ by the second way.

Note that, whenever a pair is put into the graph of *some* $\psi_{\langle \ell,0\rangle}$ in a stage of the form $2\langle p, -\rangle + 1$, that p is then put into S. Thus, there can be *no* subsequent stage of the form $2\langle p, -\rangle + 1$ in which a pair is put into the graph of *any* $\psi_{\langle \ell,0\rangle}$. It follows that, for each $\ell \in L$, the p corresponding to ℓ in the sense of (28) must be *unique* with respect to ℓ. Thus,

$$\begin{aligned}
&|\{p \mid p < p_0\}| \\
&\geq |L| && \{\text{by the preceding discussion}\} \\
&= |\{\ell \mid \mathrm{rank}(\ell) = p_0\}| && \{\text{by (27)}\} \\
&= |\{\mathrm{start}(p_0), ..., \mathrm{start}(p_0+1) - 1\}| && \{\text{by the definition of rank}\} \\
&= \mathrm{start}(p_0+1) - \mathrm{start}(p_0) && \{\text{immediate}\} \\
&= p_0 + 1 && \{\text{by the definition of start}\} \\
&= |\{p \mid p < p_0\}| + 1 && \{\text{immediate}\}
\end{aligned}$$

— a contradiction. □ (**Claim 10.1**)

Claim 10.2. 1-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$.

Proof of Claim. Follows from Claim 10.1, (23), and Lemma 1(a).
$\square$ (**Claim 10.2**)

The remainder of the proof is to show that 1-MFP does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$. By way of contradiction, let $\mu : \mathbb{N} \to \mathbb{N}$ and $(\Omega_j)_{j\in\mathbb{N}}$ witness otherwise. Let J be such that

$$J = \big\{j \mid (0,0) \in \Omega_j\big(\{(0,0)\}\big) \ \wedge\ (0,1) \in \Omega_j\big(\{(0,1)\}\big)\big\}. \tag{29}$$

Clearly, for each $j \in J$, there exist fixpoints α_0 and α_1 of Ω_j such that $(0,0) \in \alpha_0$ and $(0,1) \in \alpha_1$. Thus, for each such j, it must be that case that the *minimal* fixpoint of Ω_j diverges on 0, i.e.,

$$(\forall j \in J)[\psi_{\mu(j)}(0)\uparrow]. \tag{30}$$

Let L be such that

$$L = \{(\pi_1^2 \circ \mu)(j) \mid j \in J\}. \tag{31}$$

Claim 10.3. L is infinite.

Proof of Claim. For each i, let j_i be such that, for each α and x,

$$\Omega_{j_i}(\alpha)(x) = \begin{cases} \alpha(0), & \text{if } x = 0;\\ i, & \text{if } x = 1;\\ \uparrow, & \text{otherwise.} \end{cases} \tag{32}$$

Let L' be such that

$$L' = \{(\pi_1^2 \circ \mu)(j_i) \mid i \in \mathbb{N}\}. \tag{33}$$

Clearly, $\{j_0, j_1, ...\} \subseteq J$, and, thus, $L' \subseteq L$. To show the claim, it then suffices to show that L' is infinite. Note that, for each i and x,

$$\psi_{\mu(j_i)}(x) = \begin{cases} i, & \text{if } x = 1;\\ \uparrow, & \text{otherwise.} \end{cases} \tag{34}$$

Thus, for each i, $|\text{rng}(\psi_{\mu(j_i)})| = 1$. But, if L' were finite, then by (23), Lemma 1(c), and the pigeon-hole principle, there would exist an i such that $|\text{rng}(\psi_{\mu(j_i)})| > 1$ — a contradiction. $\square$ (**Claim 10.3**)

Let $t : \mathbb{N} \to \mathbb{N}$ be a computable, 1-1 function such that, for each j,

$$\Theta_{t(j)} = \Omega_j. \tag{35}$$

Let t^{-1} be the least partial function such that

$$t^{-1} \circ t = \text{id}. \tag{36}$$

Clearly, t^{-1} is partial computable. Thus, there exists a p_0 such that, for each k,

$$\varphi_{p_0}(k) = \begin{cases} (\mu \circ t^{-1})(k), & \text{if } k \in \text{rng}(t);\\ \uparrow, & \text{otherwise.} \end{cases} \tag{37}$$

Claim 10.4. There exists some stage of the form $2\langle p_0, -\rangle + 1$ in which each of conditions (a)-(e) in Figure 4 is satisfied.

Proof of Claim. Clearly, for each p, $\{\ell \mid \mathrm{rank}(\ell) \leq p\}$ is finite. Thus, by Claim 10.3, there exists a $j_0 \in J$ such that $\mathrm{rank}(\ell) > p_0$, where $\langle \ell, i\rangle = \mu(j_0)$. Let $k_0 = t(j_0)$. Note (a)-(e) below.

(a) $(0,0) \in \Omega_{j_0}\big(\{(0,0)\}\big) = \Theta_{k_0}\big(\{(0,0)\}\big)$.
(b) $(0,1) \in \Omega_{j_0}\big(\{(0,1)\}\big) = \Theta_{k_0}\big(\{(0,1)\}\big)$.
(c) $\varphi_{p_0}(k_0){\downarrow} = (\mu \circ t^{-1})(k_0) = \mu(j_0)$.
(d) $\psi_{\langle \ell,i\rangle}(0){\uparrow}$, where $\langle \ell, i\rangle = \varphi_{p_0}(k_0)$ {by (c) and (30)}.
(e) $\mathrm{rank}(\ell) > p_0$, where $\langle \ell, i\rangle = \varphi_{p_0}(k_0)$ {by (c) and the choice of j_0}.

Thus, for a sufficiently large s of the form $2\langle p_0, -\rangle + 1$, (a)-(e) in Figure 4 is satisfied. □ (**Claim 10.4**)

By Claim 10.4, there exists a *least* stage s_0 of the form $2\langle p_0, -\rangle + 1$ in which each of conditions (a)-(e) in Figure 4 is satisfied. Since s_0 is least such, clearly, condition (f) in Figure 4 is satisfied as well. Let k_0 be that k which is selected during stage s_0 of the construction of $(\psi_q)_{q\in N}$, let $\langle \ell, i\rangle = \varphi^{s_0}_{p_0}(k_0)$, and let $j_0 = t^{-1}(k_0)$. Note that $\langle \ell, i\rangle = \mu(j_0)$.

Claim 10.5. $(0,0) \in \psi^{s_0+1}_{\langle \ell,i\rangle}$.

Proof of Claim. Note that, by step (i) of stage s_0,

$$\big(\mathrm{out}(\langle \ell, i\rangle, 0), 0\big) \in \psi^{s_0+1}_{\langle \ell,0\rangle}. \tag{38}$$

Thus, if $i = 0$, then the claim is immediate, since $\big(\mathrm{out}(\langle \ell, i\rangle, 0), 0\big) = (0,0)$. On the other hand, if $i \geq 1$, then by step (ii) of stage s_0,

$$\psi^{s_0+1}_{\langle \ell,i\rangle} = \psi^{s_0+1}_{\langle \ell,0\rangle} \circ \mathrm{out}(\langle \ell, i\rangle, \cdot). \tag{39}$$

In particular,

$$\psi^{s_0+1}_{\langle \ell,i\rangle}(0) = \psi^{s_0+1}_{\langle \ell,0\rangle}\big(\mathrm{out}(\langle \ell, i\rangle, 0)\big). \tag{40}$$

Thus, the claim again follows from (38). □ (**Claim 10.5**)

By conditions (a) and (b) of Figure 4, $j_0 \in J$. Furthermore, by Claim 10.5,

$$(0,0) \in \psi^{s_0+1}_{\langle \ell,i\rangle} \subseteq \psi_{\langle \ell,i\rangle} = \psi_{\mu(j_0)}. \tag{41}$$

But this contradicts (30). □ (**Theorem 10**)

Theorem 11. There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$, but 1-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

Proof. Let $(\Theta_k)_{k\in\mathbb{N}}$ be any numbering of the computable operators of type $\mathcal{P} \to \mathcal{P}$. Let $(\Psi_j)_{j\in\mathbb{N}}$ be such that, for each i, k, α, and x,

$$\Psi_{\langle i,k\rangle}(\alpha)(x) = \begin{cases} \Theta_k(\alpha)(x), & \text{if } \big[[x = 0 \ \wedge \ i = 0 \ \wedge \ \alpha \neq \emptyset] \ \vee \ x > 0\big]; \\ i - 1, & \text{if } [x = 0 \ \wedge \ i > 0]; \\ \uparrow, & \text{otherwise.} \end{cases} \tag{42}$$

To show that $(\Psi_j)_{j\in\mathbb{N}}$ is exhaustive, let k be fixed, and consider the following cases.

Case $[\Theta_k(\emptyset)(0){\uparrow}]$. Then, $\Psi_{\langle 0,k\rangle} = \Theta_k$.

Case $[\Theta_k(\emptyset)(0){\downarrow}]$. Let $y = \Theta_k(\emptyset)(0)$. Then, $\Psi_{\langle y+1,k\rangle} = \Theta_k$.

Let $(\psi_q)_{q\in\mathbb{N}}$ be such that, for each q, ψ_q is the minimal fixpoint of Ψ_q. Clearly, $(\psi_q)_{q\in\mathbb{N}}$ is an eps, and 1-MFP holds in $(\psi_q)_{q\in\mathbb{N}}$ (with $\mu = \mathrm{id}$). To show that 1-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$, by way of contradiction, suppose otherwise. Then, there exists a ψ-program $\langle i,k\rangle$ such that, for each x,

$$\psi_{\langle i,k\rangle}(x) = \begin{cases} 0, & \text{if } [x = 0 \ \wedge\ i = 0]; \\ i, & \text{if } [x = 0 \ \wedge\ i > 0]; \\ \Theta_k(\emptyset)(x) + 1, & \text{if } [x > 0 \ \wedge\ \Theta_k(\emptyset)(x){\downarrow}]; \\ \uparrow, & \text{otherwise.} \end{cases} \tag{43}$$

Consider the following cases.

Case $\big[i = 0 \ \wedge\ (\forall x > 0)[\Theta_k(\emptyset)(x){\uparrow}]\big]$. Then, since $i = 0$, $\Psi_{\langle i,k\rangle}(\emptyset)(0){\uparrow}$. Furthermore, by the latter part of the case, $\Psi_{\langle i,k\rangle}(\emptyset) = \emptyset$. Thus, since $\psi_{\langle i,k\rangle}$ is the minimal fixpoint of $\Psi_{\langle i,k\rangle}$, it must be the case that $\psi_{\langle i,k\rangle} = \emptyset$. But, by (43), $\psi_{\langle i,k\rangle}(0) = 0$ — a contradiction.

Case $\big[i = 0 \ \wedge\ (\exists x > 0)[\Theta_k(\emptyset)(x){\downarrow}]\big]$. Let x and y be such that $x > 0$ and $\Theta_k(\emptyset)(x){\downarrow} = y$. Then, $\Psi_{\langle i,k\rangle}(\emptyset)(x){\downarrow} = y$. Thus, since $\psi_{\langle i,k\rangle}$ is the minimal fixpoint of $\Psi_{\langle i,k\rangle}$, it must be the case that $\psi_{\langle i,k\rangle}(x) = y$. But, by (43), $\psi_{\langle i,k\rangle}(x) = y + 1$ — a contradiction.

Case $[i > 0]$. Then, $\Psi_{\langle i,k\rangle}(\emptyset)(0){\downarrow} = i - 1$. Thus, since $\psi_{\langle i,k\rangle}$ is the minimal fixpoint of $\Psi_{\langle i,k\rangle}$, it must be the case that $\psi_{\langle i,k\rangle}(0) = i-1$. But, by (43), $\psi_{\langle i,k\rangle}(0) = i$ — a contradiction. □ (**Theorem 11**)

The proof of Theorem 13 just below is a finite-injury priority argument.

Theorem 13. For each $n \geq 1$, there exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that n-krt holds in $(\psi_q)_{q\in\mathbb{N}}$, but $(n+1)$-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

Proof. Let $n \geq 1$ be fixed. For each $i \leq n$, let α_i be such that, for each $e_0, ..., e_n$ and x,

$$\alpha_i(\langle e_0, ..., e_n, x\rangle) = \begin{cases} i, & \text{if } [|\{e_0, ..., e_n\}| < n + 1]; \\ e_{i+1}, & \text{if } [|\{e_0, ..., e_n\}| = n + 1 \ \wedge\ i < n]; \\ e_0, & \text{otherwise.} \end{cases} \tag{44}$$

Clearly, each of $\alpha_0, ..., \alpha_n$ is partial computable. Thus, to prove the theorem, it suffices to exhibit an eps $(\psi_q)_{q\in\mathbb{N}}$ in which n-krt holds, and satisfying: there is *no* $\langle q_0, ..., q_n\rangle$ such that, for each $i \leq n$,

$$\psi_{q_i} = \alpha_i(\langle q_0, ..., q_n, \cdot\rangle). \tag{45}$$

Perform stages $s = 0, 1, ...,$ successively, as follows.

STAGE $s = \langle p, - \rangle$. Let $\mathcal{Q} \subseteq 2^{\mathbb{N}}$ be the collection of all Q satisfying (a) and (b) below.

(a) Q is an $(n+1)$-cycle in $(\psi_q^s)_{q \in \mathbb{N}}$.
(b) $Q \cap \{q \mid q < n \cdot \mathrm{start}(p)\} \neq \emptyset$.

Let $k \in \{k \mid \mathrm{start}(p) \leq k < \mathrm{start}(p+1)\}$ be *least* such that

$(*)$ $\{q \mid n \cdot k \leq q < n \cdot (k+1)\} \cap \bigcup \mathcal{Q} = \emptyset$.

(Claim 13.1 shows that such a k necessarily exists.) For each $i < n$, set $\psi^{s+1}_{n \cdot k + i} = \varphi^{s+1}_{\pi^n_{i+1}(p)}(\langle n \cdot k, n \cdot k + 1, ..., n \cdot (k+1) - 1, \cdot \rangle)$.

Fig. 5. The construction of $(\psi_q)_{q \in \mathbb{N}}$ in the proof of Theorem 13.

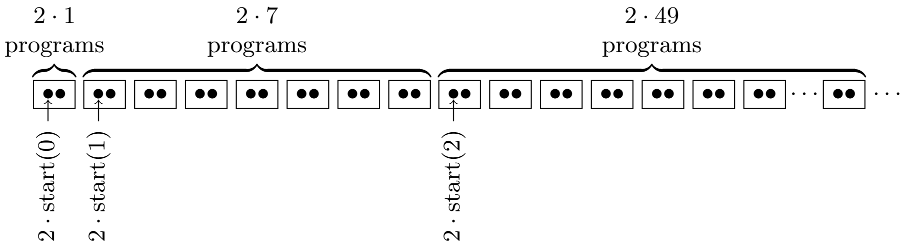

Fig. 6. A sketch of how ψ-programs are organized. Shown for the case when $n = 2$.

$(\psi_q)_{q \in \mathbb{N}}$ is constructed in stages. It will be clear from the construction that, for each stage s, the graph of $(\psi_q^s)_{q \in \mathbb{N}}$ is finite.

For each p, let start : $\mathbb{N} \to \mathbb{N}$ be as follows.

$$\mathrm{start}(0) = 0; \tag{46}$$

$$\mathrm{start}(p+1) = \mathrm{start}(p) + (n^2 + n) \cdot \mathrm{start}(p) + 1. \tag{47}$$

For each Q, n, and $(\psi'_q)_{q \in \mathbb{N}}$, we say that Q *is an* $(n+1)$-*cycle in* $(\psi'_q)_{q \in \mathbb{N}} \overset{\text{def}}{\Leftrightarrow}$ there exist $q_0, ..., q_n$ such that (a)-(d) below.

(a) For each $i, i' \leq n$, $i \neq i' \;\Rightarrow\; q_i \neq q_{i'}$.
(b) $Q = \{q_0, ..., q_n\}$.
(c) For each $i < n$, $\psi'_{q_i}(0) = q_{i+1}$.
(d) $\psi'_{q_n}(0) = q_0$.

The construction of $(\psi_q)_{q \in \mathbb{N}}$ is given in Figure 5. A sketch of how ψ-programs are organized is given in Figure 6.

Claim 13.1. For each stage $s = \langle p, - \rangle$, there exists a $k \in \{k \mid \mathrm{start}(p) \leq k < \mathrm{start}(p+1)\}$ such that k satisfies $(*)$ in Figure 5 in stage s.

Proof. Let $s = \langle p, - \rangle$ be fixed, and let $\mathcal{Q}$ be as in Figure 5 in stage s. Clearly, the $Q \in \mathcal{Q}$ are pairwise disjoint. Thus, by condition (b) in the choice of $\mathcal{Q}$,

$$|\mathcal{Q}| \leq n \cdot \text{start}(p). \tag{48}$$

By the definition of an $(n + 1)$-cycle, each $Q \in \mathcal{Q}$ consists of $n + 1$ elements. Combining this with (48), one obtains

$$|\bigcup \mathcal{Q}| \leq (n^2 + n) \cdot \text{start}(p). \tag{49}$$

Finally, by the definition of start,

$$|\{k \mid \text{start}(p) \leq k < \text{start}(p + 1)\}| = (n^2 + n) \cdot \text{start}(p) + 1. \tag{50}$$

Thus, there must exist *at least one* $k \in \{k \mid \text{start}(p) \leq k < \text{start}(p + 1)\}$ such that k satisfies $(*)$ in Figure 5 in stage s. ☐ (**Claim 13.1**)

Claim 13.2. For each p, there exists a k such that, for each $i < n$, $\psi_{n \cdot k+i} = \varphi_{\pi^n_{i+1}(p)}(\langle n \cdot k, n \cdot k + 1, ..., n \cdot (k + 1) - 1, \cdot \rangle)$.

Proof of Claim. Follows from Claim 13.1 and the construction of $(\psi_q)_{q \in \mathbb{N}}$. ☐ (**Claim 13.2**)

Claim 13.3. $(\psi_q)_{q \in \mathbb{N}}$ is an eps in which n-krt holds.

Proof of Claim. Follows from Claim 13.2. ☐ (**Claim 13.3**)

Claim 13.4. For each p, *at most one* k is selected in *infinitely many* stages of the form $\langle p, - \rangle$.

Proof. Let p, k_0, and k_1, be such that $k_0 \neq k_1$, k_0 is selected in some stage of the form $\langle p, - \rangle$, and k_1 is selected in some subsequent stage of the form $\langle p, - \rangle$. Since p, k_0, and k_1 were chosen arbitrarily, it suffices to show that k_0 is never again selected. Since k_1 is *not* selected in the earlier stage, it must be the case that $k_0 < k_1$. Since k_0 is *not* selected in the later stage, there must exist some $Q \in \mathcal{Q}$ (where $\mathcal{Q}$ is as in the later stage) such that $\{q \mid n \cdot k_0 \leq q < n \cdot (k_0 + 1)\} \cap Q \neq \emptyset$. Clearly, the existence of this Q prevents k_0 from ever again being selected. ☐ (**Claim 13.4**)

Claim 13.5. There is *no* $\langle q_0, ..., q_n \rangle$ such that, for each $i \leq n$,

$$\psi_{q_i} = \alpha_i(\langle q_0, ..., q_n, \cdot \rangle). \tag{51}$$

Proof of Claim. By way of contradiction, let $\langle q_0, ..., q_n \rangle$ witness otherwise. Let $Q = \{q_0, ..., q_n\}$. First, consider the case that $|Q| < n + 1$. Let i and i' be such that $i \neq i'$ and $q_i = q_{i'}$. Then, by (44), for each x,

$$\psi_{q_i}(x) = i \neq i' = \psi_{q_{i'}}(x) \tag{52}$$

— a contradiction. Thus, it must be the case that $|Q| = n + 1$. Furthermore, by (44), for each $i \leq n$, and each x,

$$\psi_{q_i}(x) = \begin{cases} q_{i+1}, & \text{if } i < n; \\ q_0, & \text{otherwise.} \end{cases} \tag{53}$$

Since $\text{start}(0) = 0$ and start is strictly monotone increasing, there exists a $p_{\max}$ such that

$$p_{\max} = \max\{p \mid n \cdot \text{start}(p) \leq \max Q\}. \tag{54}$$

By Claims 13.1 and 13.4, there exists a *unique* k_∞ such that k_∞ is selected in *infinitely many* stages of the form $\langle p_{\max}, -\rangle$. Let $Q_{\max}$ and Q_∞ be such that

$$Q_{\max} = \{q \mid n \cdot \text{start}(p_{\max}) \leq q < n \cdot \text{start}(p_{\max} + 1)\}; \tag{55}$$
$$Q_\infty = \{q \mid n \cdot k_\infty \leq q < n \cdot (k_\infty + 1)\}. \tag{56}$$

Clearly, by the construction of $(\psi_q)_{q\in\mathbb{N}}$, for each $q \in Q_{\max} - Q_\infty$, ψ_q is a finite function. Furthermore, by (53), for each $q \in Q$, ψ_q is a total function. Now, consider the following cases.

Case $[Q \subseteq Q_{\max}]$. Since $|Q| = n + 1 > n = |Q_\infty|$, there must exist *at least one* $q \in Q$ such that $q \notin Q_\infty$. Thus, for at least one $q \in Q$, ψ_q is a finite function — a contradiction.

Case $[Q \not\subseteq Q_{\max}]$. Clearly, for some s_0, Q is an $(n+1)$-cycle in $(\psi_q^{s_0})_{q\in\mathbb{N}}$. Furthermore, by the case, $Q \cap \{q \mid q < n \cdot \text{start}(p_{\max})\} \neq \emptyset$. Thus, $Q \in \mathcal{Q}$ in stage s_0. Furthermore, since $\psi_{\max Q}$ is a total function, $\max Q \in Q_\infty$. Thus, k_∞ does *not* satisfy $(*)$ in Figure 5 in stage s_0, or in any subsequent stage — a contradiction.

□ (**Claim 13.5**)

□ (**Theorem 13**)

The proof of Theorem 14 just below is a finite-injury priority argument.

Theorem 14. There exists an eps $(\psi_q)_{q\in\mathbb{N}}$ such that 1-KRT holds in $(\psi_q)_{q\in\mathbb{N}}$, but 2-krt does *not* hold in $(\psi_q)_{q\in\mathbb{N}}$.

Proof. The proof is essentially a combination of the techniques used in the proofs of Theorems 10 and 13. For each $i \leq 1$, let α_i be such that, for each e_0, e_1, and x,

$$\alpha_i(\langle e_0, e_1, x\rangle) = \begin{cases} i, & \text{if } [\pi_1^2(e_0) = \pi_1^2(e_1)]; \\ e_1, & \text{if } [\pi_1^2(e_0) \neq \pi_1^2(e_1) \ \wedge\ i = 0]; \\ e_0, & \text{otherwise.} \end{cases} \tag{57}$$

Clearly, α_0 and α_1 are each partial computable. Thus, to prove the theorem, it suffices to exhibit an eps $(\psi_q)_{q\in\mathbb{N}}$ in which 1-KRT holds, and satisfying: there is *no* $\langle q_0, q_1\rangle$ such that, for each $i \leq 1$,

$$\psi_{q_i} = \alpha_i(\langle q_0, q_1, \cdot\rangle). \tag{58}$$

$(\psi_q)_{q\in\mathbb{N}}$ is constructed in stages. It will be clear from the construction that, for each stage s, the graph of $(\psi_q^s)_{q\in\mathbb{N}}$ is decidable. Throughout the construction, the following invariant is maintained. (Recall that $\text{out} : \mathbb{N}^2 \to \mathbb{N}$ was defined in (8) and (9).) For each s, ℓ, and i,

$$\psi^s_{\langle \ell, i\rangle} = \psi^s_{\langle \ell, 0\rangle} \circ \text{out}(\langle \ell, i\rangle, \cdot). \tag{59}$$

Perform stages $s = 0, 1, ...,$ successively, as follows.

STAGE $s = \langle p, - \rangle$. Let $\mathcal{Q} \subseteq 2^{\mathbb{N}}$ be the collection of all Q satisfying (a) and (b) below.

(a) Q is a 2-cycle* in $(\psi_q^s)_{q \in \mathbb{N}}$.
(b) $\pi_1^2(Q) \cap \{\ell \mid \ell < \text{start}(p)\} \neq \emptyset$.

Let $\ell \in \{\ell \mid \text{start}(p) \leq \ell < \text{start}(p+1)\}$ be *least* such that

$(*)$ $\ell \notin \bigcup\{\pi_1^2(Q) \mid Q \in \mathcal{Q}\}$.

(Claim 14.2 shows that such an ℓ necessarily exists.) Set $\psi_{\langle \ell, 0\rangle}^{s+1} = \varphi_p^{s+1}$, and, for each $i \geq 1$, set $\psi_{\langle \ell, i\rangle}^{s+1} = \psi_{\langle \ell, 0\rangle}^{s+1} \circ \text{out}(\langle \ell, i\rangle, \cdot)$.

Fig. 7. The construction of $(\psi_q)_{q \in \mathbb{N}}$ in the proof of Theorem 14.

For each p and ℓ, let start : $\mathbb{N} \to \mathbb{N}$ and rank : $\mathbb{N} \to \mathbb{N}$ be as follows.

$$\text{start}(0) = 0; \tag{60}$$

$$\text{start}(p+1) = \text{start}(p) + 2 \cdot \text{start}(p) + 1. \tag{61}$$

$$\text{rank}(\ell) = \max\{p \mid \text{start}(p) \leq \ell\}. \tag{62}$$

For each Q and $(\psi'_q)_{q \in \mathbb{N}}$, we say that Q *is a 2-cycle* in* $(\psi'_q)_{q \in \mathbb{N}}$ $\overset{\text{def}}{\Leftrightarrow}$ there exist q_0 and q_1 such that (a)-(f) below.[18]

(a) $\pi_1^2(q_0) \neq \pi_1^2(q_1)$.
(b) $Q = \{q_0, q_1\}$.
(c) $\psi'_{q_0}(0) = q_1$.
(d) $\psi'_{q_1}(0) = q_0$.
(e) $\text{rng}(\psi'_{q_0}) \subseteq \{q_1\}$.
(f) $\text{rng}(\psi'_{q_1}) \subseteq \{q_0\}$.

Claim 14.1. Suppose that Q and Q' are each a 2-cycle* in $(\psi'_q)_{q \in \mathbb{N}}$. Then, either $Q = Q'$ or $\pi_1^2(Q) \cap \pi_1^2(Q') = \emptyset$.

Proof. Let Q and Q' be as stated. By way of contradiction, suppose that $Q \neq Q'$ and $\pi_1^2(Q) \cap \pi_1^2(Q') \neq \emptyset$. By the former, clearly, $Q \cap Q' = \emptyset$. Let ℓ, i, i', q, and q' be such that $Q = \{\langle \ell, i\rangle, q\}$ and $Q' = \{\langle \ell, i'\rangle, q'\}$. Thus,

$$\psi'_{\langle \ell, i\rangle}(0) = q \neq q' = \psi'_{\langle \ell, i'\rangle}(0). \tag{63}$$

Without loss of generality, suppose that $i < i'$. Then, by (59) and Lemma 1(c), $\text{rng}(\psi'_{\langle \ell, i\rangle}) \supseteq \text{rng}(\psi'_{\langle \ell, i'\rangle}) \ni q'$. But since Q is a 2-cycle* in $(\psi'_q)_{q \in \mathbb{N}}$, $\text{rng}(\psi'_{\langle \ell, i\rangle}) \subseteq \{q\} \not\ni q'$ — a contradiction. $\square$ (**Claim 14.1**)

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[18] Recall that an $(n+1)$-cycle was defined in the proof of Theorem 13. (a)-(d) imply: Q is a 2-cycle in $(\psi'_q)_{q \in \mathbb{N}}$.

The construction of $(\psi_q)_{q\in\mathbb{N}}$ is given in Figure 7.

Claim 14.2. For each stage $s = \langle p, -\rangle$, there exists an $\ell \in \{\ell \mid \text{start}(p) \le \ell < \text{start}(p+1)\}$ such that ℓ satisfies $(*)$ in Figure 7 in stage s.

Proof. Let $s = \langle p, -\rangle$ be fixed, and let $\mathcal{Q}$ be as in Figure 7 in stage s. By Claim 14.1, the sets in $\{\pi_1^2(Q) \mid Q \in \mathcal{Q}\}$ are pairwise disjoint. Thus, by condition (b) in the choice of $\mathcal{Q}$,

$$|\{\pi_1^2(Q) \mid Q \in \mathcal{Q}\}| \le \text{start}(p). \tag{64}$$

By the definition of a 2-cycle*, for each $Q \in \mathcal{Q}$, $\pi_1^2(Q)$ consists of 2 elements. Combining this with (64), one obtains

$$|\bigcup\{\pi_1^2(Q) \mid Q \in \mathcal{Q}\}| \le 2 \cdot \text{start}(p). \tag{65}$$

Finally, by the definition of start,

$$|\{\ell \mid \text{start}(p) \le \ell < \text{start}(p+1)\}| = 2 \cdot \text{start}(p) + 1. \tag{66}$$

Thus, there must exist *at least one* $\ell \in \{\ell \mid \text{start}(p) \le \ell < \text{start}(p+1)\}$ such that ℓ satisfies $(*)$ in Figure 7 in stage s. ☐ (**Claim 14.2**)

Claim 14.3. For each p, there exists an ℓ such that $\psi_{\langle \ell, 0\rangle} = \varphi_p$.

Proof of Claim. Follows from Claim 14.2 and the construction of $(\psi_q)_{q\in\mathbb{N}}$. ☐ (**Claim 14.3**)

Claim 14.4. $(\psi_q)_{q\in\mathbb{N}}$ is an eps in which 1-KRT holds.

Proof of Claim. Follows from Claim 14.3, (59), and Lemma 1(a). ☐ (**Claim 14.4**)

Claim 14.5. There is *no* $\langle q_0, q_1\rangle$ such that, for each $i \le 1$,

$$\psi_{q_i} = \alpha_i(\langle q_0, q_1, \cdot\rangle). \tag{67}$$

Proof of Claim. By way of contradiction, let $\langle q_0, q_1\rangle$ witness otherwise. First, consider the case that $\pi_1^2(q_0) = \pi_1^2(q_1)$. Then, by (57), for each x,

$$\psi_{q_0}(x) = 0 \ne 1 = \psi_{q_1}(x). \tag{68}$$

But (59), (68), and Lemma 1(c) lead to a contradiction. Thus, it must be the case that $\pi_1^2(q_0) \ne \pi_1^2(q_1)$. Furthermore, by (57), for each x,

$$\psi_{q_0}(x) = q_1; \tag{69}$$

$$\psi_{q_1}(x) = q_0. \tag{70}$$

Clearly,

$$\text{for all but finitely many } s,\ \{q_0, q_1\} \text{ is a 2-cycle* in } (\psi_q^s)_{q\in\mathbb{N}}. \tag{71}$$

Let $\ell_0 = \pi_1^2(q_0)$ and $\ell_1 = \pi_1^2(q_1)$. Without loss of generality, suppose that $\ell_0 < \ell_1$. Since $\mathrm{start}(0) = 0$ and start is strictly monotone increasing, there exists a $p_{\max}$ such that

$$p_{\max} = \max\{p \mid \mathrm{start}(p) \leq \ell_1\}. \tag{72}$$

Let $L_{\max}$ be such that

$$L_{\max} = \{\ell \mid \mathrm{start}(p_{\max}) \leq \ell < \mathrm{start}(p_{\max} + 1)\}. \tag{73}$$

Consider the following cases.

CASE $[\{\ell_0, \ell_1\} \subseteq L_{\max}]$. Then, since ψ_{q_0} and ψ_{q_1} are total, it must be the case that ℓ_0 and ℓ_1 are each selected in infinitely many stages. Since $\ell_0 < \ell_1$ and ℓ_1 is selected in infinitely many stages, the following must be true. For infinitely many stages of the form $\langle p_{\max}, - \rangle$, there exists $Q \in \mathcal{Q}$ such that $\ell_0 \in \pi_1^2(Q)$. Thus, for each such stage s and corresponding Q, (a) and (b) below.

(a) Q is a 2-cycle* in $(\psi_q^s)_{q \in \mathbb{N}}$.
(b) $\pi_1^2(Q) \cap \{\ell \mid \ell < \mathrm{start}(p_{\max})\} \neq \emptyset$.

By the case and (b) just above, $Q \neq \{q_0, q_1\}$. Furthermore, since $\ell_0 \in \pi_1^2(Q)$, $\pi_1^2(Q) \cap \pi_1^2(\{q_0, q_1\}) \neq \emptyset$. Thus, this case and (71) contradict Claim 14.1.

CASE $[\{\ell_0, \ell_1\} \not\subseteq L_{\max}]$. Then, $\pi_1^2(\{q_0, q_1\}) \cap \{\ell \mid \ell < \mathrm{start}(p_{\max})\} \neq \emptyset$. Furthermore, by (71), ℓ_1 is selected in *at most* finitely many stages. Thus, ψ_{q_1} is a finite function — a contradiction. $\square$ (**Claim 14.5**)

$\square$ (**Theorem 14**)